MATHEMATICS

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Abstract

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THE PROBLEM OF CHEBYSHEV APPROXIMATION IN A COMMUTATIVE HILBERT RING

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1. Let $\varphi_1(q), \ldots, \varphi_n(q)$ be functions continuous on some compact set Q, with values in an infinite-dimensional commutative Hilbert ring H. Any element $a \in H$ has the form $a = \sum_\alpha a_\alpha e_\alpha$, where the coefficients a_α are complex numbers, and $\{e_\alpha\}$ is a basis of mutually orthogonal irreducible Hermitian idempotents (^{1}). The functions $\varphi_k(q)$ $(k = 1, 2, \ldots, n)$ have the form

$$\varphi_k(q) = \sum_\alpha \varphi_{\alpha k}(q) e_\alpha,$$

where $\varphi_{\alpha k}(q)$ are complex-valued functions continuous on Q.

We shall approximate, in the best possible way, a function $f(q)$, continuous on Q and with values in H, by means of polynomials of the form $\sum_{k=1}^n a_k \varphi_k(q)$, where $a_1, \ldots, a_n$ are elements of H, i.e., we shall seek among them such a polynomial $\sum_{k=1}^n a_k^{(0)} \varphi_k(q)$ for which

$$\max_{q\in Q} \left\| \sum_{k=1}^n a_k^{(0)} \varphi_k(q) - f(q) \right\| = \inf_{a_k \in H} \max_{q\in Q} \left\| \sum_{k=1}^n a_k \varphi_k(q) - f(q) \right\|.$$

The case of a finite-dimensional ring was considered in (^{2}).

2. Denote by T the set of all complexes $a = (a_1, \ldots, a_n)$, where $a_k \in H$ $(k = 1, 2, \ldots, n)$, for which

$$\sum_{k=1}^n a_k \varphi_k(q) \equiv \theta$$

on Q.

It is clear that T is a subspace of the Hilbert space H^n with norm

$$\|a\|_{H^n} = \left(\sum_{k=1}^{n} \|a_k\|^2 \right)^{1/2}.$$

Let S denote the orthogonal complement of T in H^n.

We note that approximation by means of the polynomial $\sum_{k=1}^{n} a_k \varphi_k(q)$ may be regarded as approximation by means of the operator-function $A(q)$, which for each $q \in Q$ is a linear bounded operator acting from H^n into H according to the formula

$$A(q)a = \sum_{k=1}^{n} a_k \varphi_k(q),$$

and from (^{3,4}) we obtain that, for the existence for every continuous function of a polynomial

for least deviation it is necessary and sufficient that the condition

$$\max_{q \in Q} \left\| \sum_{k=1}^{n} a_k \varphi_k(q) \right\| \geq m \left(\sum_{k=1}^{n} \|a_k\|^2 \right)^{1/2} \quad \text{for all } (a_1, \dots, a_n) \in S, \tag{a}$$

be satisfied, where $m > 0$ is a constant. From this condition there follows the following theorem.

Theorem 1. *In order that for every function $f(q)$, continuous on Q and with values in H, there exist a polynomial of least deviation, it is necessary and sufficient that the subspace S be finite-dimensional, or, what is the same, that each of the functions*

$$\varphi_k(q) = \sum_{\alpha} \varphi_{\alpha k}(q) e_\alpha \quad (k = 1, 2, \dots, n)$$

have only a finite number of coefficients $\varphi_{\alpha k}(q)$ that are not identically equal to zero on Q.

3. Denote by L the set of indices α such that $\varphi_{\alpha k}(q) \not\equiv 0$ on Q for at least one $k = 1, 2, \dots, n$. The number l of such indices, by the preceding theorem, is finite, and $\dim S \leq nl$.

Theorem 2. *If the dimension of the subspace S is a multiple of l: $\dim S = tl$, then, in order that for every continuous function $f(q)$ there exist a unique polynomial of least deviation from it*

$$\sum_{k=1}^{n} a_k^{(0)} \varphi_k(q),$$

where $(a_1^{(0)}, \dots, a_n^{(0)}) \in S$*, it is necessary and sufficient that every polynomial*

$$\sum_{k=1}^{n} a_k \varphi_k(q),$$

for which $\sum_{k=1}^{n} \|a_k\| > 0$ *and* $(a_1, \dots, a_n) \in S$*, vanish at no more than* $t-1$ *points of the compact set* Q*.*

The proof of this theorem can be obtained by considering the polynomial

$$\sum_{k=1}^{n} a_k \varphi_k(q) \quad ((a_1, \dots, a_n) \in S)$$

as an operator-function acting from a finite-dimensional space into a finite-dimensional one, as in (4).

The case in which H is finite-dimensional and $\dim S = n \dim H$ was considered in (2).

Remark. If $\dim S$ is not a multiple of l, then the question of uniqueness becomes more complicated, and, in addition to restrictions on the number of zeros of the polynomials

$$\sum_{k=1}^{n} a_k \varphi_k(q)$$

$$\left(\sum_{k=1}^{n} \|a_k\| > 0 \right),$$

additional restrictions in the spirit of (5) are also needed.

4. Let us now suppose that the functions $\varphi_1(q), \dots, \varphi_n(q)$, continuous on Q and with values in H, are such that the corresponding subspace S is infinite-dimensional; in particular, $S = H^n$ (in this latter case the functions $\varphi_1(q), \dots, \varphi_n(q)$ are "linearly independent" on Q in the sense that

$$\sum_{k=1}^{n} a_k \varphi_k(q) \equiv 0$$

only when $a_1 = \dots = a_n = 0$).

Noting that S is always separable (even when H is nonseparable), we shall, for convenience, assume H separable and $S = H^n$.

By Theorem 1, now not for every function $f(q)$, continuous on Q and with values in H, does there exist a polynomial of least deviation. Denote by F_φ the set of those functions, continuous on Q and with values in H, for which such polynomials exist. We note that F_φ is dense in $C(H, Q)$—the Banach space of all functions continuous on Q and with values in H.

Theorem 3. *Let the continuous functions* $\varphi_1(q), \dots, \varphi_n(q)$ *be such that every polynomial*

$$\sum_{k=1}^{n} a_k \varphi_k(q) \left(\sum_{k=1}^{n} \|a_k\| > 0 \right)$$

vanishes at no more than $n-1$ *points of the compact set* Q *(which contains more than* n *points).*

Then, in order that the polynomial $\sum_{k=1}^{n} a_k^{(0)} \varphi_k(q)$ deviate least on Q from the function $f(q) \in F_\varphi$, it is necessary that the deviation

$$\max_{q \in Q} \left\| \sum_{k=1}^{n} a_k^{(0)} \varphi_k(q) - f(q) \right\|$$

be attained at no fewer than $n+1$ points of the compact set Q.

Theorem 4. In order that for every function $f(q) \in F_\varphi$ there exist a unique polynomial of least deviation, it is necessary and sufficient that every polynomial $\sum_{k=1}^{n} a_k \varphi_k(q)$ $\left(\sum_{k=1}^{n} \|a_k\| > 0\right)$ vanish at no more than $n-1$ points of the compact set Q. This condition is equivalent to requiring that, for each $\alpha = 1, 2, \dots$, the numerical functions $\varphi_{\alpha 1}(q), \varphi_{\alpha 2}(q), \dots, \varphi_{\alpha n}(q)$, where $\varphi_k(q) = \sum_{\alpha=1}^{\infty} \varphi_{\alpha k}(q) e_\alpha$ $(k = 1, 2, \dots, n)$, form a Chebyshev system on Q.

5. As follows from Theorem 1, in order that for every function $f(q)$, continuous on Q and with values in H, when approximating it by H-functions $a\varphi(q)$, there exist a function $a^{(0)}\varphi(q)$ of least deviation, one has to impose a very restrictive condition on the function $\varphi(q) = \sum_\alpha \varphi_\alpha(q) e_\alpha$ (namely, that only a finite number of the coefficients $\varphi_\alpha(q)$ be not identically equal to zero on Q).

If, however, as the approximating function one takes not the function $a\varphi(q)$, but a function of the form $\lambda a - a\varphi(q)$, where $\lambda \neq 0$ is some complex number, then, as established in (6), from the continuity alone of the function $\varphi(q)$ there already follows the existence of a function $\lambda a^{(0)} - a^{(0)}\varphi(q)$ of least deviation for every continuous function $f(q)$, and in order that for any continuous function $f(q)$ there exist a unique function $\lambda a^{(0)} - a^{(0)}\varphi(q)$ $(a^{(0)} \in H)$ of least deviation, it is necessary and sufficient that, for every $a \neq \theta$ in H, the function $\lambda a - a\varphi(q)$ nowhere vanish on Q, which is equivalent to the condition that for each α at all points q of the compact set Q one have $\varphi_\alpha(q) \neq \lambda$.

Thus, whatever the continuous function $\varphi(q)$, choosing λ so that $|\lambda| > \max_{q\in Q}\|\varphi(q)\|$ ensures uniqueness of the function of least deviation for any continuous function being approximated.

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Note: Figure translations are in progress. See original paper for figures.

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