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Abstract

Full Text

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Saturation Classes for Certain Summation Processes

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1°. Let E be a Banach space in which there is a closed system of elements $\{f_k\}$ $(k = 1, 2, ...)$ and a complete sequence of functionals $\{l_k(f)\}$ $(k = 1, 2, ...)$, forming a biorthogonal system, i.e.

$$l_k(f_i) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

To each element $f \in E$ one can associate the expansion

$$f \sim \sum_{k=1}^{\infty} l_k(f) f_k. \tag{1}$$

In the case of convergence of this series, its sum will be f. The summation process (Δ) of the series (1), consisting in the formation of the means

$$u_n^{\lambda}(f) = \sum_{k=1}^{\infty} \lambda_k^{(n)} l_k(f) f_k, \tag{2}$$

where the multipliers $\lambda_k^{(n)}$ $(n = 1, 2, ..., \ k = 1, 2, ...)$ are prescribed numbers (which may have as an upper index a continuous parameter h), is called **saturated** if the means (2) cannot approach f too rapidly. More precisely, there exist a positive function $\varphi(n)$ monotonically tending to zero and a number m such that the relation

$$\|f - u_n^{\lambda}(f)\| = o[\varphi(n)]$$

is satisfied only for elements of the form $f = a_1 f_1 + a_2 f_2 + \cdots + a_{mf} m$, where $a_1, a_2, \ldots, a_m$ are numbers.

If, in addition, there exists an element f not expressible linearly in terms of $f_1, f_2, \ldots, f_m$ and for which the relation

$$\|f - u_n^{\lambda}(f)\| = O[\varphi(n)] \tag{3}$$

holds, then the order of saturation of the process (Δ) is $\varphi(n)$. The linear set of elements satisfying relation (3) is called the **saturation class** of the process (Δ).

The saturation problem for the process (Δ) consists in finding the saturation class of this process. To find the saturation class means to find the structural properties of the functions satisfying relation (3).

The formulation of the problem in the form in which it is set forth here belongs to Favard $({}^{1,2})$. The solution of this problem for certain summation processes

···mation was given by Zamansky $({}^{3})$. Butzer $({}^{4})$ characterized the classes of saturation of some processes by means of the theory of semigroups.

In the present paper I find several classes of saturation by an elementary method, which was indicated by Favard in $({}^{1})$.

2°. Let E be one of the spaces of periodic functions: $C(-\pi,\pi)$, $L_p(-\pi,\pi)$ $(p \geq 1)$, and let the expansion (1) be the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

Theorem 1. *If the numbers* $\lambda_k^{(n)}$ $(k = 0,1,2,\dots,\ n = 1,2,\dots)$ *satisfy the condition*

$$\lim_{n\to\infty} \frac{1-\lambda_k^{(n)}}{\varphi(n)} = \gamma_k < \infty,$$

then for every function $f(x)$ *satisfying the relation*

$$\|f(x) - u_n^{\lambda}(f,x)\| = O[\varphi(n)], \tag{4}$$

where

$$u_n^{\lambda}(f,x) = \frac{a_0}{2}\lambda_0^{(n)} + \sum_{k=1}^{\infty}\lambda_k^{(n)}(a_k \cos kx + b_k \sin kx),$$

the Fejér means of the series

$$\frac{a_0}{2}\gamma_0 + \sum_{k=1}^{\infty}\gamma_k(a_k \cos kx + b_k \sin kx)$$

are bounded in norm in the space E.

Proof. From (4) it follows that the norm of the function

$$F(x) = \frac{1}{\varphi(n)}[f(x) - u_n^{\lambda}(f,x)]$$

does not exceed some constant A. Write the expansion

$$F(x) \sim \frac{a_0}{2}\frac{1-\lambda_0^{(n)}}{\varphi(n)} + \sum_{k=1}^{\infty}\frac{1-\lambda_k^{(n)}}{\varphi(n)}(a_k\cos kx + b_k\sin kx),$$

and consider the Fejér means for this series:

$$\sigma_m(F,x) = \frac{a_0}{2}\frac{1-\lambda_0^{(n)}}{\varphi(n)} + \sum_{k=1}^{m}\left(1-\frac{k}{m}\right)\frac{1-\lambda_k^{(n)}}{\varphi(n)}(a_k\cos kx + b_k\sin kx).$$

It is known that for every m, $\|\sigma_m(F,x)\| \le \|F\|$. Consequently,

$$\left\|\frac{a_0}{2}\frac{1-\lambda_0^{(n)}}{\varphi(n)} + \sum_{k=1}^{m}\left(1-\frac{k}{m}\right)\frac{1-\lambda_k^{(n)}}{\varphi(n)}(a_k\cos kx + b_k\sin kx)\right\| \le A.$$

Letting n tend to infinity, we obtain from this

$$\left\|\frac{a_0}{2}\gamma_0 + \sum_{k=1}^{m}\left(1-\frac{k}{m}\right)\gamma_k(a_k\cos kx + b_k\sin kx)\right\| \le A.$$

This proves the theorem.

Let us apply the theorem just proved to known summation processes.

3°. For the Bernstein-Rogozinski process (BR,ν),

$$\lambda_k^{(n)} = \left(\cos\frac{k\pi}{2n-1}\right)^{\nu}$$

for $k = 0, 1, \dots, n$ and $\lambda_k^{(n)} = 0$ for $k > n$. The order of saturation of this process is $\frac{1}{n^2}$, and $\gamma_k = \frac{\pi^2}{8}\nu k^2$. Consequently, if for some function $f \in E$

$$\|f(x) - B_n^{(\nu)}(f,x)\| = O\left(\frac{1}{n^2}\right),$$

where

$$B_n^{(\nu)}(f,x) = \frac{a_0}{2} + \sum_{k+1}^{n}\left(\cos\frac{k\pi}{2n+1}\right)^{\nu}(a_k\cos kx + b_k\sin kx),$$

then

$$\left\| \frac{a_0}{2} + \sum_{k=1}^{m} \left(1 - \frac{k}{m}\right) k^2 (a_k \cos kx + b_k \sin kx) \right\| = O(1). \tag{5}$$

If (5) holds for the space $C(-\pi, \pi)$, then almost everywhere there exists a bounded second derivative $f''(x)$, or, what is the same, $f'(x)$ satisfies a Lipschitz condition. The converse assertion is known in the case $\nu = 1$ (5). The general case is easily obtained from this by means of formula (4) from (6).

Thus, we have:

Theorem 2. In order that a continuous periodic function $f(x)$ satisfy the relation

$$\max_x |f(x) - B_n^{(\nu)} f(x)| = O\left(\frac{1}{n^2}\right),$$

it is necessary and sufficient that $f'(x) \in \mathrm{Lip}\, 1$.

Consequently, the saturation classes of all processes (BP, ν) coincide. Applying the known results (7) on classes of trigonometric series and relation (5), we obtain the following theorems.

Theorem 3. The saturation class of the process (BP, ν) in the space L_p, for any ν, coincides with the class of functions equivalent to absolutely continuous functions having second derivatives belonging to the space L_p.

Theorem 4. The saturation class of the process (BP, ν) in the space L_1, for any ν, coincides with the class of functions equivalent to absolutely continuous functions whose first-order derivatives have bounded variation.

4°. Let us consider the process of summing a Fourier series by the so-called typical means:

$$X_n^{(\nu)}(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(1 - \frac{k^\nu}{n^\nu}\right) (a_k \cos kx + b_k \sin kx).$$

Here $\lambda_k^{(n)} = 1 - \dfrac{k^\nu}{n^\nu}$ for $k = 0, 1, \dots, n$ and $\lambda_k^{(n)} = 0$ for $k > n$. The order of saturation is $\dfrac{1}{n^\nu}$ and $\gamma_k = k^\nu$. Consequently, according to Theorem 1, if for a continuous function $f(x)$

$$\max_x |f(x) - X_n^{(\nu)}(x)| = O\left(\frac{1}{n^\nu}\right), \tag{6}$$

then

$$\max_x \left| \frac{a_0}{2} + \sum_{k=1}^{m} \left(1 - \frac{k}{m}\right) k^{\nu} (a_k \cos kx + b_k \sin kx) \right| = O(1).$$

From this condition, for even ν it follows that a bounded $f^{(\nu)}(x)$ exists almost everywhere. For odd ν, the conjugate function $\bar{f}(x)$ has a bounded derivative of order ν almost everywhere. The converse assertions were proved by Zygmund (5).

Thus we have:

Theorem 5. *In order that a continuous periodic function $f(x)$ satisfy relation (6), it is necessary and sufficient that there exist a bounded derivative $f^{(\nu)}(x)$ for even ν, and a bounded derivative $\bar{f}^{(\nu)}(x)$ for odd ν.*

For $\nu = 1$ the process under consideration coincides with the process $(C, 1)$, and Theorem 5 reduces to the theorem of Alexits–Zamansky (8,3).

Analogous theorems are also valid for the space L_p $(p \geqslant 1)$.

5°. Finally, let us consider the Lebesgue summation process, defined by the multipliers $\lambda_k^h = \dfrac{\sin kh}{kh}$. For a summable function, the Lebesgue means coincide with the Steklov functions (7, p. 269):

$$f_h(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{\sin kh}{kh} (a_k \cos kx + b_k \sin kx).$$

It is easy to see that this process is saturated and that the order of saturation is h^2. Moreover,

$$\gamma_k = \lim_{h \to \infty} \frac{1}{h^2} \left(1 - \frac{\sin kh}{kh}\right) = \frac{k^2}{6}.$$

Consequently, according to Theorem 1, if $\|f_h(x) - f(x)\| = O(h^2)$, then (5) is fulfilled.

Thus we have:

Theorem 6. *The saturation class of the Lebesgue process (or of the process of approximation by Steklov means) in the space G or L_p $(p \geqslant 1)$ coincides with the saturation class of the process $(\ , 1)$ in the corresponding space.*

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Note: Figure translations are in progress. See original paper for figures.