Mathematics

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Abstract

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INVERSE THEOREMS OF THE CONSTRUCTIVE THEORY OF FUNCTIONS OF SEVERAL VARIABLES

(Presented by Academician A. N. Kolmogorov, 13 II 1958)

The purpose of the present note is to generalize to the case of functions of several variables the following result:

Theorem. Let $f(x)$ be a periodic function of period 2π. Then

$$\Delta_h^r f(x) = O\left\{n^{-r}\sum_{k=1}^{n} k^{r-1}E_{k-1}(f)\right\}, \qquad h = O\left(\frac{1}{n}\right), \tag{1}$$

where $E_n = E_n(f)$ is the best approximation of the function $f(x)$ by trigonometric polynomials of order $\leq n$ in the metric of the corresponding space (C or L_p);

$$\Delta_h^r f(x) = \sum_{i=0}^{r}(-1)^{r-i}C_r^i f(x+ih), \qquad C_r^i = \frac{r(r-1)\dots(r-i+1)}{i!}.$$

Inequality (1) was first obtained in ([1]) for the metric L_p, and was then extended ([2,5]) to the case of the uniform metric.

For a function $f(x,y)$ periodic of period 2π in each variable, an analogous theorem holds:

$$\left\|\Delta_{h_1}^{r_1}\Delta_{h_2}^{r_2} f(x,y)\right\| \leq \frac{C}{m^{r_1}n^{r_2}}\sum_{k=1}^{m}\sum_{l=1}^{n} k^{r_1-1}l^{r_2-1}E_{k-1,l-1}, \tag{2}$$

where

$$E_{k,l} = E_{k,l}(f) = \inf_T \|f(x,y) - T_{k,l}(x,y)\|; \qquad h_1 = O\left(\frac{1}{m}\right), \quad h_2 = O\left(\frac{1}{m}\right),$$

$T_{m,n}(x,y)$ is a trigonometric polynomial of order m in x, of order n in y;

$$\Delta_{h_1}^{r_1}\Delta_{h_2}^{r_2}f(x,y)=\sum_{i=0}^{r_1}\sum_{j=0}^{r_2}(-1)^{r_1+r_2-i-j}C_{r_1}^{i}C_{r_2}^{j}f(x+ih_1,y+jh_2).$$

Considering the special case $r_1=r_2=2$ under the assumption that

$$E_{m,n}=O\left(\frac{1}{mn}\right),$$

Dzhvaršeishvili ([3]) proved that

$$\Delta_{h_1}^{2}\Delta_{h_2}^{2}f(x,y)=O(h_1+h_2). \tag{3}$$

From inequality (2), under the same assumptions, it follows that

$$\Delta_{h_1}^{2}\Delta_{h_2}^{2}f(x,y)=O(h_1\cdot h_2).$$

However, it should be noted that if

$$E_{m,n}(f)=O\left(\frac{1}{(m+1)(n+1)}\right),$$

then the function $f(x,y)\equiv\text{const}$ $(m,n=0,1,2,...)$.

Proof of inequality (2). Let $\{T_{mn}(x,y)\}$ be a sequence of trigonometric polynomials giving, for each m and n, the best approximation to the function $f(x,y)$. Suppose that $2^p<m\le 2^{p+1}$, $2^q<n\le 2^{q+1}$; then, obviously,

$$\left\|\Delta_{h_1}^{r_1}\Delta_{h_2}^{r_2}f(x,y)-\Delta_{h_1}^{r_1}\Delta_{h_2}^{r_2}T_{2^{p+1},\,2^{q+1}}\right\|\le 2^{r_1+r_2}E_{2^p,\,2^q}. \tag{4}$$

To prove inequality (2), we estimate:

$$\left\|\frac{\partial^{r_1+r_2}}{\partial x^{r_1}\partial y^{r_2}}T_{2^{p+1},\,2^{q+1}}(x,y)\right\| = \left\|T^{(r_1+r_2)}_{2^{p+1},\,2^{q+1}}\right\|$$

$$\le \left\|T^{(r_1+r_2)}_{2,2}\right\| + \sum_{k=1}^{p-1}\left\|T^{(r_1+r_2)}_{2^{k+1},\,2} - T^{(r_1+r_2)}_{2^k,\,2}\right\|$$

$$+ \sum_{l=1}^{q-1}\left\|T^{(r_1+r_2)}_{2,\,2^{l+1}} - T^{(r_1+r_2)}_{2,\,2^l}\right\|$$

$$+ \left\|T^{(r_1+r_2)}_{2^{p+1},\,2^{q+1}} - T^{(r_1+r_2)}_{2^p,\,2^q}\right\|$$

$$+ \sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\left\|T^{(r_1+r_2)}_{2^{k+1},\,2^{l+1}} - T^{(r_1+r_2)}_{2^{k+1},\,2^l} - T^{(r_1+r_2)}_{2^k,\,2^{l+1}} + T^{(r_1+r_2)}_{2^k,\,2^l}\right\|.$$

Using the well-known inequality of S. N. Bernstein for derivatives of trigonometric polynomials and the obvious inequalities

$$\left\|T_{2^{k+1},\,2} - T_{2^k,\,2}\right\| \le 2E_{2^k,\,2} \le 2^{2r_1+1}\cdot 2^{-(k+1)r_1}\sum_{i=2^{k-1}+1}^{2^k} i^{r_1-1}E_{i,\,2}; \tag{5}$$

$$\left\|T_{2^{k+1},\,2^{l+1}} - T_{2^{k+1},\,2^l} - T_{2^k,\,2^{l+1}} + T_{2^k,\,2^l}\right\| \le 4E_{2^k,\,2^l} \le 2^{2r_1+2r_2+2}\frac{\sum\limits_{i=2^{k-1}+1}^{2^k}\sum\limits_{j=2^{l-1}+1}^{2^l} i^{r_1-1}j^{r_2-1}E_{ij}}{2^{(k+1)r_1}\cdot 2^{(l+1)r_2}}, \tag{6}$$

we obtain

$$\begin{aligned}\left\|T^{(r_1+r_2)}_{2^{p+1},\,2^{q+1}}\right\| \le\ & 2^{r_1+r_2+1}E_{0,0} + 2^{2r_1+r_2+1}\sum_{k=1}^{p-1}\sum_{i=2^{k-1}+1}^{2^k} i^{r_1-1}E_{i,\,2}\\ & + 2^{2r_2+r_1+1}\sum_{l=1}^{q-1}\sum_{j=2^{l-1}+1}^{2^l} j^{r_2-1}E_{2,\,j}\\ & + 2^{2r_1+2r_2+1}\sum_{k=2^{p-1}+1}^{2^p}\sum_{l=2^{q-1}+1}^{2^q} k^{r_1-1}l^{r_2-1}E_{k,\,l}\\ & + 2^{2r_1+2r_2+2}\sum_{k=1}^{p-1}\sum_{l=1}^{q-1}\sum_{i=2^{k-1}+1}^{2^k}\sum_{j=2^{l-1}+1}^{2^l} i^{r_1-1}j^{r_2-1}E_{i,\,j}.\end{aligned} \tag{7}$$

Inequality (7) gives

$$\left\| T^{(r_1+r_2)}_{2^{p+1},\,2^{q+1}} \right\| = O\left\{ \sum_{k=1}^{m} \sum_{l=1}^{n} k^{r_1-1} l^{r_2-1} E_{k-1,\,l-1} \right\}. \tag{8}$$

From inequalities (4), (6), and (8) it follows that

$$\left\| \Delta^{r_1}_{h_1} \Delta^{r_2}_{h_2} f(x,y) \right\| = O\left\{ \frac{1}{m^{r_1} n^{r_2}} \sum_{k=1}^{m} \sum_{l=1}^{n} k^{r_1-1} l^{r_2-1} E_{k,\,l} + h_1^{r_1} h_2^{r_2} \left\| T^{(r_1+r_2)}_{2^{p+1},\,2^{q+1}} \right\| \right\}.$$

Taking $h_1 = O\left(\frac{1}{m}\right)$ and $h_2 = O\left(\frac{1}{n}\right)$, we obtain (2).

For a periodic function of k variables $f(x_1, \dots, x_k)$, the same method makes it possible to prove the inequality

$$\left\| \Delta^{r_1}_{h_1} \cdots \Delta^{r_k}_{h_k} f(x_1, \dots, x_k) \right\| =$$

$$= O\left\{ n_1^{-r_1} \cdots n_k^{-r_k} \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} i_1^{r_1-1} \cdots i_k^{r_k-1} E_{i_1-1,\dots,i_k-1} \right\},$$

$$h_\nu = O\left(\frac{1}{n_\nu}\right) \quad (\nu = 1, \dots, k).$$

We shall also state, without proof, the following theorem:

Theorem. If, for a continuous function $f(x_1, \dots, x_m)$, the series

$$\sum_{n_1=1}^{\infty} n_1^{r-1} E_{n_1,\dots,n_k,\infty}(f) \quad (k \leqslant m),$$

converges, where $n_i = [n_1^{\sigma_i}]$ $(0 < \sigma_i \leqslant 1;\ i = 2, 3, \dots, k)$; $E_{n_1,\dots,n_k,\infty}(f)$ is the partial best uniform approximation of the function $f(x_1, \dots, x_k)$ by trigonometric polynomials of degree $\leqslant n_j$ in the variables x_j $(j = 1, 2, \dots, k)$ with coefficients that are continuous functions of the variables $(x_{k+1}, \dots, x_m)$, then the function $f(x_1, \dots, x_m)$ has a continuous mixed derivative

$$\frac{\partial^p f(x_1, \dots, x_m)}{\partial x_1^{p_1} \cdots \partial x_k^{p_k}},$$

where $p_1 + \dots + p_k = p$,

$$p_1 + \sum_{j=2}^{k} \sigma_j p_j = r.$$

For $k = m = 2$, $E_{n_1,n_2} = O\left(\frac{1}{n_1^{\alpha}} + \frac{1}{n_2^{\beta}}\right)$, $\sigma_2 = \frac{\alpha}{\beta}$ $(\alpha \leqslant \beta)$, we obtain, as a special case, Montel' s theorem ([4]).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.