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Abstract

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SYMMETRIC ENERGY INEQUALITIES AND THE MIXED PROBLEM

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The present article is devoted to the construction of a generalized solution of the mixed problem for a general hyperbolic equation of second order with right-hand side a functional of a certain class. For the case when the coefficients do not depend on time, a similar problem was considered by means of the generalized Laplace transform in (^{1}). Our method uses ideas developed in (^{2}) for the Cauchy problem. In an analogous way one can study the ordinary parabolic equation and equations of the types introduced in (^{3}) (the mixed problem), and also strengthen the results obtained in (^{2}) for the Cauchy problem. The constructions carried out are naturally connected with a number of considerations in the papers ($^{4-7}$).

Let Ω be a bounded star-shaped domain in the space of variables $(x_1, \dots, x_\nu)$, and let $V = [0 \le x_0 \le 1] \times \Omega$. Consider in V the linear hyperbolic operator, written in the form

$$a = \sum_{k,l=0}^{\nu} D_k(a_{kl} D_l + a_k) + c, \qquad D_k \equiv \frac{\partial}{\partial x_k}, \tag{1}$$

where $a_{kl} = a_{lk}$; all coefficients are assumed bounded and summable in V, and, in addition, a_k $(k = 0, \dots, \nu)$ have bounded derivatives either with respect to x_k or with respect to x_0, the coefficients a_{0k} $(k = 1, \dots, \nu)$ have bounded derivatives with respect to x_k, and the remaining coefficients a_{kl} have continuous derivatives with respect to x_0. Thus the notation (1) is conditional in character, and the exact definition of the operator a is given below. Together with the operator a we shall consider the operators

$$a^* \equiv \sum_{l,k=0}^{\nu} D_l(a_{kl} D_k - a_l) + c^*,$$

$$b \equiv D_0; \qquad b^* \equiv -D_0.$$

If a_k has no derivatives with respect to x_k, then the corresponding group of terms of a^* has the form $\sum a_{kD}k$, and some modification of the constructions presented is necessary. If $\dot{\Omega}$ is the boundary of Ω, then the classical form of the simplest boundary conditions associated with a is

$$u\big|_{x_0=0} = D_0 u\big|_{x_0=0} = 0, \tag{Γ_{n}}$$

$$u\big|_{S_\delta} = 0, \qquad S_\delta = [0 \le x_0 \le 1] \times \dot{\Omega}. \tag{Γ_{k}}$$

The totality of the initial and boundary conditions will be denoted by (Γ). The conditions (Γ^*) are obtained by replacing (Γ_{n}) with

$$v\big|_{x_0=1} = D_0 v\big|_{x_0=1} = 0. \tag{Γ^*_{n}}$$

In V we consider the Hilbert space $\mathcal{H}$ of functions with summable square, with the usual inner product and norm, denoted by

$$(u,\ v,\ V) = \int_V uv\,dV, \qquad |u, \mathcal{H}|^2 = (u, u, V),$$

and the classes of functions $\mathcal{E}_1, \mathcal{E}_2$ from C^∞, satisfying respectively the conditions (Γ) or (Γ^*). For $u \in \mathcal{E}_1$, $v \in C^\infty$ and when the conditions (Γ_k) are fulfilled, we set, by definition:

$$(au,\ v,\ V_t) = (D_0 a_{00} D_0 u,\ v,\ V_t) -- \sum_{k,l=0}^{\nu}{}'(a_{kl} D_{lu} + a_{ku},\ D_{kv},\ V_t) + (cu,\ v,\ V_t), \tag{2}$$

where $V_t = [0 \le x_0 \le t] \times \Omega$, and the notation $(\dots, \dots, V_t)$ means that integration is carried out over the corresponding part of the volume V. In $\sum'$ the term written first is absent. Then, in particular, for $u \in \mathcal{E}_1$

$$(au,\ bu,\ V_t) = \int_{S_t} \left[a_{00} D_0 u D_0 u - \sum_{k,l=0}^{\nu}{}' a_{kl} D_{luD_{ku}} \right] dS_t + \int_{V_t} F(u,u)\,dV_t \equiv$$

$$\equiv \int_{S_t} F_0'(u,u)\,dS_t + \int_{V_t} F(u,u)\,dV_t, \tag{3}$$

where S_t is the part of the plane $x_0 = t$ lying in V, and $F(u,u)$ is an inhomogeneous form in the function u and its derivatives, with bounded coefficients formed by $a_k,\ c$, the derivatives of a_{kl} with respect to x_0, and of a_{0k} with respect to x_k. Let

$$|Du,\ S_t|^2 = \int_{S_t} \sum_{k=0}^{\nu} |D_{ku}|^2\, dS_t.$$

By hyperbolicity of the operator a we understand the fact that there exists a constant $C > 0$ such that

$$|Du,\ S_t|^2 \le C \int_{S_t} F_0'(u,u)\, dS_t$$

for all $t \in [0,1]$. For $u \in C^\infty$ set

$$|u,\ B^1| = \sup_{t\in[0,1]} |Du,\ S_t|.$$

The closure of $\mathcal{E}_\sigma$ $(\sigma = 1,2)$ in the metric B^1 gives the Banach space B^1_σ. Together with B^1_σ we consider the space $B^{1,t}_\sigma$ of functions h such that $h \in B^1_\sigma,\ D_0 h \in B^1_\sigma$. Let us further note that if to the operator b^* (b) one adjoins initial conditions of the form $h|_{x_0=0} = 0$ $(h|_{x_0=1} = 0)$, then an operator b^{*-1} (b^{-1}) is uniquely defined, given, for example, on all $u \in \mathcal{H}$. We shall say that $u \in \mathfrak{B}_1$ $(u \in \mathfrak{B}^*_1)$ if $b^{*-1}u \in B^1_1$ $(B^{1,t}_1)$. We want to define the operator a on an arbitrary function $u \in \mathfrak{B}_1$, considering au as a functional over some space. In view of the nonreflexivity of B^1_σ, for what follows it is convenient to introduce the Hilbert spaces W^1_σ, obtained by closing $\mathcal{E}_\sigma$ in the norm

$$|u,\ W^1|^2 = \int_0^1 |Du,\ S_t|^2\, dt.$$

Note that an arbitrary element $f \in \mathcal{H}$

can be regarded as a functional on W^1_σ, defined by the formula $\langle f, v\rangle = (f, v, V)$, with norm

$$|f, W^{-1}_\sigma| = \sup_{v\in W^{1,t}_\sigma} \frac{|\langle f,v\rangle|}{|v, W^1|} = \sup_{v\in W^1_\sigma} \frac{|\langle f,v\rangle|}{|v, W^1|}, \tag{4}$$

where $v \in W^{1,t}_\sigma$, if $v \in W^1_\sigma,\ D_0 v \in W^1_\sigma$, and the last equality is valid by virtue of the density of $W^{1,t}_\sigma$ in W^1_σ. The closure of $\mathfrak{B}$ in the norm (4) gives the Hilbert space W^{-1}_σ. For $h \in \mathfrak{E}_1,\ v \in \mathfrak{E}_2$

$$(ab^*h, v, V) = \sum_{k,l=0}^{\nu} (a_{kl}D_{lD}0h + a_{kD}0h, D_{kv}, V) - (cD_0h, v, V) =$$

$$= -\sum_{k,l=0}^{\nu}(a_{kl}D_{lh}, D_0D_{kv}, V) + \sum_{k,l=0}^{\nu}(a_{kD}0h - (D_0a_{kl})D_{lh}, D_{kv}, V)$$

$$-(cD_0h, v, V) \equiv (\tilde{a}h, \tilde{b}v, V), \tag{5}$$

where the last term denotes an abbreviated notation for the preceding expression. Now associating with every function $u \in \mathfrak{B}$ the function $h \in B_1^1$ according to the rule $u = b^*h$ $(h = b^{*-1}u)$, we put, by definition,

$$\langle au, v\rangle \equiv \langle ab^*h, v\rangle = (\tilde{a}h, \tilde{b}v, V), \qquad v \in W_2^{1,t}.$$

Thus, we have defined the functional au on a set dense in W_2^1. Then

$$|au, W_2^{-1}| = \sup_{v \in W_2^{1,t}} \frac{|(\tilde{a}h, \tilde{b}v, V)|}{|v, W^1|} = \sup_{v \in W_2^{1,t}} \frac{|\langle au, v\rangle|}{|v, W^1|}. \tag{6}$$

If the norm of au is finite, then, by continuity, it can be extended to all of W_2^1. We note that, as follows from (5), this is always so for $h \in B_1^{1,t}$. In this case (6) may be written for $v \in W_2^1$. All the constructions carried out can in the obvious way be transferred to the operator a^* with the introduction of the operator b.

We now write the generalized energy inequalities for the operators a, a^*, the proof of which is the main problem:

$$|b^{*-1}u, B^1| \leq C\,|au, W_2^{-1}|; \qquad (\ )$$

$$|b^{-1}v, B^1| \leq C\,|a^*v, W_1^{-1}|. \tag{Φ^*}$$

Since the inequalities are completely symmetric (in contrast to the dual inequalities in (2)), it is sufficient to prove, for example, (Φ).

Lemma 1. If $h \in B_1^{1,t}$, then

$$|h, B^1| \leq C \sup_{v \in W_2^1} \frac{|\langle ab^*h, v\rangle|}{|v, W^1|}.$$

The proof, following essentially the scheme of (2), uses the possibility of taking as v a solution of the equation

$$bv = \begin{cases} bh, & 0 \leq x_0 \leq t, \\ 0, & t < x_0 \leq 1; \end{cases} \qquad v\big|_{x_0=1} = 0.$$

Lemma 2. If $h \in \dot{B}_1^1$, then

$$|h, B^1| \leq C \sup_{v \in W_2^{1,t}} \frac{|\langle ab^* h, v\rangle|}{|v, W^1|}.$$

The proof uses the transfer of averaging operators with respect to x_0, which ensure the identical fulfillment of the conditions (Γ_h), (Γ_h^*) (cf. (7)).

Replacing h by $b^{*-1}u$ in Lemma 2, we obtain the inequality (Φ) for arbitrary $u \in \mathfrak{B}_1$.

We pass to the formulation and proof, following from the inequalities (Φ), (Φ^*), of the theorem on existence and uniqueness of the generalized solution of the mixed problem for the equation

$$au = f. \tag{7}$$

Lemma 3. The following inclusions hold: $W_2^1 \subset \mathfrak{B}_2$ ($v \in \mathfrak{B}_2$, if $b^{-1}v \in B_2^1$); $W_1^{1,t} \subset \mathfrak{B}_1^t$.

Lemma 4. Functionals of the form au, $u \in W_1^{1,t}$, form a set dense in W_2^{-1}.

Since the space W_2^{-1} is Hilbert, i.e. reflexive, it suffices to prove the completeness of the corresponding system of functionals. But, since $v \in W_2^1 \subset \mathfrak{B}_2$, and the function u belongs to $W_1^{1,t}$, we have $\langle au, v\rangle = \langle u, a^* v\rangle$, as follows from (5) and the corresponding definitions. Thus, the validity for v of the equality $\langle au, v\rangle = 0$ for every $u \in W_1^{1,t}$ implies, by virtue of (Φ^*),

$$\left|b^{-1}v, B^1\right| = 0. \tag{8}$$

But if $v \in W_2^1$ and (8) holds, then $\left|v, W^1\right| = 0$, q.e.d.

We now define a generalized solution for equation (7), using the closure of the operator a (with initial domain of definition $W_1^{1,t}$) in W_2^{-1}. Let $f \in W_2^{-1}$. It follows from Lemma 4 that there exists a sequence of functions $u_i \in W_1^{1,t}$ such that

$$\left|au_i - f, W_2^{-1}\right| \to 0 \quad \text{as } i \to \infty.$$

It follows from the inequality (Φ) that the sequence $b^{*-1}u_i$ converges in B_1^1. Denoting the limiting function by $b^{*-1}u$, we shall call u the **generalized solution of equation** (7) **in the strong sense**. A function u satisfying equality (7) in the sense of the original definition of the functional au will naturally be called a **weak solution of equation** (7). A strong solution is, evidently, at the same time a weak one. By virtue of the uniqueness of the weak solution, which follows from (Φ), for $f \in W_2^{-1}$ every weak solution is a strong one, which we shall call the **generalized solution of equation** (7). Thus, we obtain Theorem 1:

Theorem 1. The generalized solution of equation (7), for arbitrary $f \in W_2^{-1}$, exists, is unique, and belongs to the class $\mathfrak{B}_1$.

Theorem 2. If, in addition to the assumptions made, the coefficients of the operator a do not depend on x_0, then for $f \in \mathcal{H}$ the generalized solution u belongs to B_1^1.

For the proof it suffices to define $D_0 f$ as an element of W_2^{-1} and to consider equation (7) differentiated with respect to x_0. Theorem 2 shows that for $f \in \mathcal{H}$ and for an operator a independent of x_0, the solution defined above coincides with the usual generalized solution[6].

The chosen boundary conditions (Γ_k) are not, of course, the only possible ones.

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Note: Figure translations are in progress. See original paper for figures.

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