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Abstract

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ON THE GENERAL PROBLEM OF THE THEORY OF NEUTRON SLOWING DOWN

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The state of a neutron in the theory of slowing down is determined by specifying the corresponding phase point $Q = (\mathbf{r}, \vec{\Omega}, u)$, where $\mathbf{r}$ is the spatial coordinate of the neutron; $\mathbf{r} \in G$; G is a connected domain of three-dimensional space R_3, in which the process under consideration takes place. The complement of G to the whole space is occupied by black bodies*. $\vec{\Omega} = |\mathbf{V}|^{-1}\mathbf{V}$, where $\mathbf{V}$ is the velocity of the neutron; $\vec{\Omega} \in S$; S is the unit sphere of three-dimensional space. The lethargy

$$u = -\ln \frac{|\mathbf{V}|^2}{2E_0}; \qquad E_0 = \sup \frac{|\mathbf{V}|^2}{2},$$

where the supremum is taken over all neutrons; $u \in U_1 = [0, \infty)$. Thus, $Q \in H = G \times S \times U_1$. Slowing down occurs on nuclei of various kinds, filling G in an arbitrary manner. Here, for simplicity, it is assumed that G decomposes into parts, each of which contains nuclei of one and the same mass: $M = M(\mathbf{r})$ (the mass of the neutron is taken as unity). If $l_s(\mathbf{r}, u)$, $l_c(\mathbf{r}, u)$ are the total mean free paths for scattering and absorption, then let

$$\beta(\mathbf{r}, u) = l_s^{-1}(\mathbf{r}, u) + l_c^{-1}(\mathbf{r}, u)$$

and

$$h(\mathbf{r}, u) = \beta^{-1}(\mathbf{r}, u)\, l_s^{-1}(\mathbf{r}, u).$$

Then the phase density of neutron collisions $\psi(Q)$ satisfies the integral equation

$$\psi(Q) = \hat{A}\psi(Q) + T(Q); \tag{1}$$

$$\hat{A}F(Q) = \beta(\mathbf{r}, u) \int_0^{|\mathbf{r}-\vec{\rho}_0|} \exp\left[-\int_0^{\rho} \beta(\mathbf{r} - r'\vec{\Omega}, u)\, dr'\right] d\rho \times$$

$$\times \int_S d\vec{\Omega}' \int_0^u h(\mathbf{r} - \rho\vec{\Omega}, u')\, g(\mathbf{r} - \rho\vec{\Omega}, \vec{\Omega}\vec{\Omega}', u', u)\, F(\mathbf{r} - \rho\vec{\Omega}, \vec{\Omega}', u')\, du'. \tag{2}$$

Here $\vec{\rho}_0 = \vec{\rho}_0(\mathbf{r}, \vec{\Omega}) = \mathbf{r} - t\vec{\Omega} \in \overline{G} \setminus G$; t is the distance from $\mathbf{r}$ to the boundary $\overline{G} \setminus G$ of the domain G in the direction $-\vec{\Omega}$; $t = \inf t'$, $\mathbf{r} - t'\vec{\Omega} \in \overline{G}$; $g(\mathbf{r}, \mu, u', u)$ is the scattering indicatrix through an angle $\arccos \mu$ with a change of lethargy from the value u' to the value u.

The free term of equation (1) is determined by the boundary conditions and the distribution of sources:

$$T(Q) = \beta(\mathbf{r}, u) \int_0^{|\mathbf{r}-\vec{\rho}_0|} \exp\left[-\int_0^{\rho} \beta(\mathbf{r} - r'\vec{\Omega}, u)\, dr'\right] S(\mathbf{r} - \rho\vec{\Omega}, \vec{\Omega}, u)\, d\rho +$$

$$+ |\mathbf{V}(u)|\, \beta(\mathbf{r}, u)\, B(\vec{\rho}_0, \vec{\Omega}, u) \exp\left[-\int_0^{|\mathbf{r}-\vec{\rho}_0|} \beta(\mathbf{r} - r'\vec{\Omega}, u)\, dr'\right]. \qquad (3)$$

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* If G is convex, the "black body" may be a vacuum.

$B(\vec{\rho}_0, \vec{\Omega}, u)$ is the phase density of neutrons at a boundary point $\vec{\rho}_0 \in \bar{G} \setminus G$. The function B is defined for all $u > 0$, $\vec{\rho}_0 \in \bar{G} \setminus G$, and those $\vec{\Omega}$ which correspond to the direction from $\vec{\rho}_0$ into G; $S(Q)$ is the phase strength of the sources; $Q \in H$.

Let $H(\bar{u})$ be the part of H on which $u \in [0, \bar{u})$, and let $K(\gamma)$ be the set of all functions $F(Q)$ measurable on H such that $\exp(-\gamma|r|)F(Q)$ is bounded on every $H(\bar{u})$, $\bar{u} < \infty$. Put

$$\alpha(r) = \frac{(M(r) + 1)^2}{4M(r)}$$

and

$$\omega_+ = \sup_{Q \in H} \omega(Q), \qquad \omega_- = \inf_{Q \in H} \omega(Q),$$

for any $\omega(Q)$ defined on H.

The study of equation (1) is carried out under the following assumptions:

1. $\beta(r, u) \in K(0)$; $h(r, u) \in K(0)$; $M(r) \in K(0)$, $M_+ < \infty$.
2. $0 < \beta_- \le \beta_+ < \infty$.
3. The primary radiation has lethargy u not exceeding $u_0 \in (0, \infty]$.
4. $S(Q) \in K(\delta)$, $\delta \in [0, \beta_-)$, and

$$S^+ = \sup_{Q \in H} S(Q) e^{-\delta|r|} < \infty.$$

5. $B(\vec{\rho}_0, \vec{\Omega}, u) \le B^+ e^{\delta|\vec{\rho}_0|}$, $B^+ = \text{const}$.

The assumption $\beta_- > 0$ means that domains G not containing voids are being considered.

The indicatrix $g(r, \mu, u', u)$ is uniquely determined by the law of a single scattering and can be written, for example, as follows:

$$g(r,\mu,u',u) = \frac{\alpha(r)}{\pi} e^{-(u-u')} \Bigg\{ \sigma_0(r,\mu,u',u)\,\delta(\mu - \mu_0(r,u-u'))$$
$$+ \sum_{n=1}^{\infty} \frac{\sigma_n(r,\mu,u',u)}{\left(1 - \frac{M(r)+1}{M(r)}\frac{E_n(r)}{E_0} e^{u'}\right)^{1/2}}\,\delta\left(\mu - \mu_0(r,u-u') - \frac{M(r)E_n(r)}{2E_0} e^{(u-u')/2}\right)$$
$$+ \frac{2}{M(r)} \frac{W(r,\mu,u',u)}{\left(1 - \frac{2\mu}{M(r)+1} e^{(u-u')/2} + \frac{1}{(M(r)+1)^2} e^{u-u'}\right)^{1/2}} \Bigg\},$$
$$\mu_0(r,u) = \frac{M(r)+1}{2} e^{-u/2} - \frac{M(r)-1}{2} e^{u/2}. \qquad (4)$$

The terms in the right-hand side of (4) that contain the δ-function correspond to excitation of a nucleus of mass $M(r)$ as a result of inelastic scattering of a neutron to the level $E_n(r)$; the last term corresponds to the continuous spectrum of excitation of the nucleus. Possible anisotropy of scattering in the center-of-mass system is taken into account by the factors $\sigma_k(r, \mu, u', u)$ and $W(r, \mu, u', u)$.

As follows from (4), practically without restricting the generality of the physical formulation of the problem, one may assume that for all admissible values of the independent variables:

$$6. \quad \sigma_k(r,\mu,u',u) \le \sigma_k^0 < \infty, \quad k = 0, 1, 2, \ldots; \qquad \sum_{k=1}^{\infty} \sigma_k = \sigma < \infty.$$

$$7. \quad W(r,\mu,u',u) \le \tau < \infty.$$

The singularities appearing in (4) introduce no fundamental difficulties, since, using (4) and (2), one can give a definition of $\hat{A}$ that contains no singularities.

The study of the properties of $\hat{A}$ is greatly facilitated by the circumstance that this operator is an operator of Volterra type with respect to the variable u.* It is not difficult to show that $\hat{A}$ maps a measurable function into

* In particular, $\hat{A}$ has no eigenfunctions in the classes $K(\gamma)$, $\gamma < \beta_-$ (cf. the proof of assertion 8). The eigenvalue problem acquires meaning when the structure of $\hat{A}$ is changed, for example, when multiplication and acceleration of neutrons are included in the consideration.

measurable. Let $\gamma < \beta_-$, $p > 0$. Then, by direct calculation, the inequality can be established

$$\hat{A}e^{\gamma|\mathbf{r}|+(p-1)u} \leq \frac{\Phi(p)}{\beta_- - \gamma}\, e^{\gamma|\mathbf{r}|+(p-1)u}, \tag{5}$$

$$\Phi(p) = 2\alpha_+\beta_+h_+\left[\frac{4\tau + \sigma_0(1-e^{-pq_+})}{p} + \sigma B\left(p, \frac{1}{2}\right)\right]. \tag{6}$$

Here B is Euler' s integral of the first kind, $q_+ = \ln\left(\frac{M_+ + 1}{M_- - 1}\right)^2$.

From (5) we obtain

$$\hat{A}[K(\gamma)] \subset K(\gamma), \qquad \gamma < \beta_-.$$

8. If $F(Q) \in K(\gamma)$, $\gamma < \beta_-$, then, uniformly with respect to all $Q \in H(\bar{u})$, $\bar{u} < \infty$,

$$\lim_{\nu=\infty} e^{-\gamma|\mathbf{r}|}\hat{A}^\nu|F(Q)| = 0.$$

Indeed, for $p \geq 1$ and a suitable constant $F_{\bar{u}}$,

$$|F(Q)| \leq F_{\bar{u}}e^{\gamma|\mathbf{r}|+(p-1)u}, \qquad Q \in H(\bar{u}).$$

According to (6), p can be chosen so large that $(\beta_- - \gamma)^{-1}\Phi(p) \leq 1/2$, so that for $Q \in H(\bar{u})$

$$e^{-\gamma|\mathbf{r}|}\hat{A}^\nu|F(Q)| \leq 2^{-\nu}F_{\bar{u}}e^{(p-1)\bar{u}},$$

as was required.

Let $p > 0$, $k \in [0, \beta_- - \delta)$, $L > 0$, and let $R(\mathbf{r})$ denote the distance from the point $\mathbf{r}$ to the set of boundary points of $\bar{G}$ at each of which either the source density S or the flux through the boundary B is nonzero. Put

$$\psi_1(Q) = Le^{-kR(\mathbf{r})+\delta|\mathbf{r}|+(p-1)u}. \tag{7}$$

Then

$$\hat{A}\psi_1(Q) \leq \frac{\Phi(p)}{\beta_- - (\delta+k)}\left(1 - e^{-R_0(\beta_- - (\delta+k))}\right)\psi_1(Q), \tag{8}$$

where R_0 is the diameter of G; $R_0 \in (0, \infty]$. As follows from 3–5 and formula (3), $T(Q) \in K(\delta)$ and

$$T(Q) \leq L_1 e^{\delta|\mathbf{r}|-(\beta_- - \delta)R(\mathbf{r})} Y(u_0 - u). \tag{9}$$

Here $Y(x)$ is the Heaviside function:

$$Y(x) = 0 \text{ for } x \leq 0, \qquad Y(x) = 1 \text{ for } x > 0; \qquad L_1 = B^+ \beta_+ \sqrt{2E_0} + \frac{S^+ \beta_+}{\beta_- - \delta};$$

$$\hat{A}\psi_1(Q) + T(Q) \leq \psi_1(Q) \left\{ \frac{L_1}{L} e^{-(\beta_- - (k+\delta))R(\mathbf{r}) - (p-1)u} Y(u_0 - u) + \frac{\Phi(p)}{\beta_- - (k+\delta)} \left(1 - e^{-R_0(\beta_- - (\delta+k))}\right) \right\}. \tag{10}$$

Obviously, there exists exactly one $p = p(k)$ such that

$$(\beta_- - (k+\delta))^{-1} \Phi(p(k)) \left(1 - \exp[-R_0(\beta_- - (\delta+k))]\right) = 1.$$

Let $p > \max\{p(k), 0\}$. If $u_0 = \infty$, require in addition $p \geq 1$. Then there exists L such that the expression in braces in (10) does not exceed unity for all values $Q \in H$. Fix these p and L:

$$\hat{A}\psi_1(Q) + T(Q) \leq \psi_1(Q), \qquad Q \in H, \tag{11}$$

whence, for any nonnegative integers n and m,

$$\sum_{\nu=n}^{n+m} \hat{A}^\nu T(Q) \leq \hat{A}^n \psi_1(Q). \tag{12}$$

(12) means that the series

$$\sum_{\nu=0}^{\infty} \hat{A}^\nu T(Q) = \psi(Q)$$

converges for all $Q \in H$, and

$$\psi(Q) \leqslant \psi_1(Q), \qquad Q \in H. \tag{13}$$

Hence, $\psi(Q) \in K(\delta)$. Moreover, from 8 and (12) it follows that, uniformly with respect to all $Q \in H(\bar{u})$, $\bar{u} < \infty$,

$$\lim_{n=\infty} e^{-\delta|r|} \sum_{\nu=0}^{n} \hat{A}^\nu T(Q) = e^{-\delta|r|} \psi(Q). \tag{14}$$

Using 8 and (14), it is not difficult to show that $\psi(Q)$ is a solution of equation (1). However, this also follows immediately from (12) and Lebesgue' s theorem on dominated convergence.

Thus we obtain:

9. Under assumptions 1–7, equation (1) has exactly one solution $\psi(Q)$ in the class of functions $\bigcup\limits_{\gamma<\beta_-} K(\gamma)$. This solution belongs to $K(\delta)$, is represented by the Neumann series, and admits the estimate (13) for a suitable choice of the constants L and p depending on δ and k.

The uniqueness of the solution is an immediate consequence of 8. According to 9, in $\bigcup\limits_{\gamma<\beta_-} K(\gamma)$ the homogeneous equation (1) has only the trivial solution.

In the particular case of elastic scattering one may drop the requirement $p > 0$. For example, if G is the whole space R_3, filled with a homogeneous moderator scattering neutrons elastically and isotropically in the center-of-mass system, and the sources have a plane-parallel structure and are concentrated in a layer of finite thickness orthogonal to the Cartesian axis Oz, then the estimate (13) of the solution $\psi(Q) \equiv \psi(z, \vec{\Omega}, u)$ takes the form

$$\psi(z, \vec{\Omega}, u) \leqslant L e^{-k|z|+(p-1)u}, \tag{15}$$

$k \in [0, 1]$, $p > p^*(k)$, where

$$\frac{\alpha h}{1-k} \cdot \frac{1}{p(k)} \left(1 - e^{-qp(k)}\right) = 1, \qquad \alpha = \frac{(M+1)^2}{4M},$$

$$q = \ln \left(\frac{M+1}{M-1}\right)^2.$$

(15) is valid under the assumption that the characteristics of the medium do not depend on the neutron energy. The mean free path is taken to be unity.

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Note: Figure translations are in progress. See original paper for figures.