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Abstract

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MATHEMATICS

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ON THE DECOMPOSITION OF THE TENSOR PRODUCT OF REPRESENTATIONS OF THE PRINCIPAL SERIES OF THE PROPER LORENTZ GROUP INTO IRREDUCIBLE REPRESENTATIONS

(Presented by Academician A. N. Kolmogorov on 4 XII 1957)

The decomposition into irreducible representations of the tensor product of finite-dimensional irreducible representations of the Lorentz group is well known and is used in theoretical physics; for infinite-dimensional representations the analogous question has not yet been considered. The present note is devoted to the solution of this question for the tensor product of representations of the principal series of the proper Lorentz group.

Throughout this note we use the notation and results of ([1,2]). In addition, as usual, instead of representations of the Lorentz group we consider representations of the locally isomorphic group $\mathfrak{G}$ of second-order matrices with determinant equal to one.

1. Let us recall that representations of the principal series can be realized in the Hilbert space $L^2(Z)$ of functions $f(z)$, $z = x + iy$, satisfying the condition* $\int |f(z)|^2 \, dz < \infty$. Representations of the principal series are given by two parameters m, σ, where m is an integer and σ is a real number, and the representation corresponding to m, σ (we denote it by $\mathfrak{S}_{m,\sigma}$) is defined by the formula

$$V_g f(z) = |\beta z + \delta|^{-m+i\sigma-2} (\beta z + \delta)^m f\left(\frac{\alpha z + \gamma}{\gamma z + \delta}\right) \quad \text{for} \quad g = \begin{vmatrix} \alpha & \beta \\ \beta & \delta \end{vmatrix}.$$

The tensor product $\mathfrak{S}_{m_1,\sigma_1} \times \mathfrak{S}_{m_2,\sigma_2}$ of the representations $\mathfrak{S}_{m_1,\sigma_1}$, $\mathfrak{S}_{m_2,\sigma_2}$ is defined as the representation $g \to T_g$ in the Hilbert space $L^2(Z \times Z)$ of functions $f(z_1, z_2)$ satisfying the condition $\int |f(z_1, z_2)| \, dz_1 dz_2 < \infty$; in this case

$$T_g f(z_1, z_2) = |\beta z_1 + \delta|^{-m_1 + i\sigma_1 - 2} (\beta z_1 + \delta)^{m_1} |\beta z_2 + \delta|^{-m_2 + i\sigma_2 - 2} \beta(z_2 + \delta)^{m_2} \times$$

$$\times f\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right).$$

Let U be the operator in $L^2(Z \times Z)$ defined by the formula

$$U f(z_1, z_2) = |z_2|^{-m_2 + i\sigma_2 - 2} z_2^{m_2} f\left(z_1, z^1 + \frac{1}{z_2}\right).$$

It is easy to verify that U is a unitary operator in $L^2(Z \times Z)$ and that the operator $T_g' = U T_g U^{-1}$ is given by the formula

$$T_g' f(z_1, z_2) = |\beta z_1 + \delta|^{-m_1 + m_2 + i(\sigma_1 - \sigma_2)} (\beta z_1 + \delta)^{m_1 - m_2} \times$$
$$\times f\left[\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, (\beta z_1 + \delta)^2 z_2 + (\beta z_1 + \delta)\beta\right]. \tag{1}$$

Thus:

I. *Formula (1) defines a representation* $g \to T_g'$, *unitarily equivalent to the representation* $\mathfrak{S}_{m_1,\sigma_1} \times \mathfrak{S}_{m_2,\sigma_2}$.

It follows from this:

II. *Two representations* $\mathfrak{S}_{m_1,\sigma_1} \times \mathfrak{S}_{m_2,\sigma_2}$, $\mathfrak{S}_{m_1',\sigma_1'} \times \mathfrak{S}_{m_2',\sigma_2'}$, *for which*

$$m_1 - m_2 = m_1' - m_2' \quad \text{and} \quad \sigma_1 - \sigma_2 = \sigma_1' - \sigma_2',$$

are unitarily equivalent.

2. Using assertion I, one can obtain the decomposition of the regular representation of the group $\mathfrak{G}$ into the representations $\mathfrak{S}_{m_1,\sigma_1} \times \mathfrak{S}_{m_2,\sigma_2}$. Recall that the regular representation $g \to W_g$ is a representation in the Hilbert space $L^2(\mathfrak{G})$ of functions $f(g)$ which satisfy the condition $\int |f(g)|^2\, dg < \infty$, with $W_{g_0} f(g) = f(g g_0)$.

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* In what follows, dz, for any complex variable $z = x + iy$, denotes $dx\, dy$, and the integral with respect to this variable is taken over the whole complex plane; further,
$\int f(z_1, z_2, \dots, z_k)\, dz_1 dz_2 \cdots dz_k$ denotes integration with respect to each of the variables $z_1, z_2, \dots, z_k$ over the whole complex plane.

Put

$$f(z, \zeta, \chi) = \int x(\delta_\zeta z) \chi(\lambda) \frac{d\lambda}{|\lambda|^2}, \tag{2}$$

where

$$\delta = \begin{Vmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{Vmatrix}, \qquad \zeta = \begin{Vmatrix} 1 & \zeta \\ 0 & 1 \end{Vmatrix}, \qquad z = \begin{Vmatrix} 1 & 0 \\ z & 1 \end{Vmatrix}, \tag{3}$$

and $\chi(\lambda) = |\lambda|^{-m+i\sigma} \lambda^m$ is a character of the group of all matrices δ. Then

$$\int |x(g)|^2 \, dg = \int |f(z, \zeta, \chi)|^2 \, dz \, d\zeta \, d\chi, \tag{4}$$

where $d\chi = \dfrac{1}{4\pi^2} \, d\sigma$ and the integral with respect to $d\chi$ is taken over the whole group X of characters χ.

On passing from $f(g)$ to $f(gg_0)$, the function $f_\chi(z, \zeta) = f(z, \zeta, \chi)$, for each fixed $\chi(\lambda) = |\lambda|^{-m+i\sigma} \lambda^m$, is transformed according to the representation $g \to T'_g$, in which $m = m_1 - m_2$, $\sigma = \sigma_1 - \sigma_2$.

Consequently:

III. *Formulas (2) and (4) carry out the decomposition of the regular representation of the group* $\mathfrak{G}$ *into the representations* $\mathfrak{S}_{m_1, \sigma_1} \times \mathfrak{S}_{m_2, \sigma_2}$.

3. Denote by X_0 the totality of all characters $\chi(\lambda) = |\lambda|^{-m+i\sigma} \lambda^m$ such that $m - m_1 + m_2$ is an even number. Further, denote by $\mathfrak{M}$ the Hilbert space of all measurable functions $f(z, \chi) = f(z, m, \sigma)$ such that:

a)

$$\int_{X_0} \omega(\chi) \, d\chi \int |f(z, \chi)|^2 \, dz < \infty, \qquad \text{where } \omega(\chi) = \frac{1}{4\pi^2} (m^2 + \sigma^2);$$

b) the Fourier transform with respect to z,

$$\hat{f}(w, \chi) = \frac{1}{2\pi} \int f(z, \chi) e^{-i \operatorname{Re}(z\overline{w})} \, dz,$$

of the function $f(z, \chi)$ satisfies the condition

$$\hat{f}(w, \chi^{-1}) = (-i)^m 2^{-i\sigma} |w|^{m+i\sigma} w^{-m} \times \tag{5}$$
$$\times \frac{\Gamma\left(-\frac{m+m_1-m_2}{2} + \frac{1}{2} - i\frac{\sigma+\sigma_1-\sigma_2}{2}\right) \Gamma\left(\frac{m-m_1+m_2}{2} + \frac{1}{2} - i\frac{\sigma-\sigma_1+\sigma_2}{2}\right)}{\Gamma\left(-\frac{m+m_1-m_2}{2} + \frac{1}{2} + i\frac{\sigma+\sigma_1-\sigma_2}{2}\right) \Gamma\left(\frac{m-m_1+m_2}{2} + \frac{1}{2} + i\frac{\sigma-\sigma_1+\sigma_2}{2}\right)} \hat{f}(w, \chi).$$

for almost all $\omega, \chi;\ \chi \in X_0$, while the scalar product in $\mathfrak{M}$ is defined by the formula

$$(f_1, f_2) = \int_{X_0} \omega(\chi)\, d\chi \int f_1(z, \chi)\overline{f_2(z, \chi)}\, dz.$$

The following main theorem holds:

Theorem. For every function $f(z_1, z_2) \in L^2(Z \times Z)$ the integral

$$f(z, \chi) = \int f(z_1, z_2) a(z_1, z_2, z, \chi)\, dz_1 dz_2, \tag{6}$$

where

$$\begin{aligned} a(z_1, z_2, z, \chi) = {} & |z_2 - z_1|^{\frac{m+m_1+m_2}{2} - i\frac{\sigma+\sigma_1+\sigma_2}{2} - 1} (z_2 - z_1)^{-\frac{m+m_1+m_2}{2}} \times \\ & \times |z - z_1|^{-\frac{m-m_1+m_2}{2} + i\frac{\sigma-\sigma_1+\sigma_2}{2} - 1} (z - z_1)^{\frac{m-m_1+m_2}{2}} \times \\ & \times |z_2 - z|^{-\frac{m+m_1-m_2}{2} + i\frac{\sigma+\sigma_1-\sigma_2}{2} - 1} (z_2 - z)^{\frac{m+m_1-m_2}{2}}, \end{aligned} \tag{7}$$

converges in the sense of the norm in $\mathfrak{M}$, and the correspondence $f(z_1, z_2) \to f(z, \chi)$, defined by formula (6), is an isometric mapping $Sf(z_1, z_2) = f(z, \chi)$ of the space $L^2(Z \times Z)$ onto the space $\mathfrak{M}$; the inverse mapping $S^{-1} = S^*$ is given by the formula

$$S^{-1} f(z, \chi) = \int_{X_0} d\chi \int f(z, \chi)\overline{a(z_1, z_2, z, \chi)}\, dz.$$

Under passage from $f(z_1, z_2) \in L^2(Z \times Z)$ to

$$T_g f(z_1, z_2) = |\beta z_1 + \delta|^{-m_1 + i\sigma_1 - 2} (\beta z_1 + \delta)^{m_1} |\beta z_2 + \delta|^{-m_2 + i\sigma_2 - 2} \times$$

$$\times (\beta z_2 + \delta)^{m_2} f\left(\frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}, \frac{\alpha z_2 + \gamma}{\beta z_2 + \delta}\right)$$

the function $f(z, \chi) = Sf(z_1, z_2)$ passes into

$$|\beta z + \delta|^{-m + i\sigma - 2} (\beta z + \delta)^m f\left(\frac{\alpha z + \gamma}{\beta z + \delta}, \chi\right);$$

consequently, the mapping S realizes the decomposition of the representation $\mathfrak{S}_{m_1, \sigma_1} \times \mathfrak{S}_{m_2, \sigma_2}$ into irreducible representations of the principal series.

In the proof of this theorem one uses the Plancherel formula (3) for the group $\mathfrak{G}$ and Proposition III.

Remark 1. Let X_0^+ denote the totality of all characters $\chi \in X_0^+$ such that $m > 0$, or $m = 0$ and $\sigma \geqslant 0$. In view of condition (4), the space $\mathfrak{M}$ can be realized as the totality of all measurable functions $f(z, \chi)$, $\chi \in X_0^+$, such that

$$\int_{X_0^+} \omega(\chi)\, d\chi \int |f(z,\chi)|^2\, dz < \infty,$$

with scalar product

$$(f_1, f_2) = 2\int_{X_0^+} \omega(\chi)\, d\chi \int f_1(z,\chi)\overline{f_2(z,\chi)}\, dz.$$

The assertion of the preceding theorem will hold also under this realization of the space $\mathfrak{M}$.

4. Recall (see (1), § 6) that the representations $\mathfrak{S}_{m,\sigma}$ can also be realized in the Hilbert space $L^2_m(\mathfrak{U})$ of all measurable functions

$\tilde{f}(u)$ on the unitary subgroup $\mathfrak{U}$ of the group $\mathfrak{G}$ such that $\int |\tilde{f}(u)|^2 du < \infty$, $\tilde{f}(\gamma u) = \alpha(\gamma)\tilde{f}(u)$, where $\alpha(g) = |g_{22}|^{-m+i\sigma-2} g_{22}^m$; moreover

$$V_g \tilde{f}(u) = \frac{\alpha(ug)}{\alpha(\overline{ug})}\tilde{f}(ug),$$

where $u\overline{g}$ denotes the matrix u_1 defined by the relation $ug = \delta\zeta u_1$. The transition to this realization is given by the formula

$$\tilde{f}(u) = \sqrt{\pi\alpha(u)}\, f(z) \quad \text{for } u = \delta\zeta,$$

where δ, ζ, z are matrices of the form (3). In accordance with this, the representation $\mathfrak{S}_{m_1,\sigma_1} \times \mathfrak{S}_{m_2,\sigma_2}$ is realized in the space $L^2_{m_1}(\mathfrak{U}) \times L^2_{m_2}(\mathfrak{U})$ of measurable functions $\tilde{f}(u_1, u_2)$ such that

$$\iint |\tilde{f}(u_1,u_2)|^2\, du_1 du_2 < \infty, \quad \tilde{f}(\gamma_1 u_1, \gamma_2 u_2) = \alpha_1(\gamma_1)\alpha_2(\gamma_2)\tilde{f}(u_1,u_2),$$

where

$$\alpha_1(g) = |g_{22}|^{-m_1+i\sigma_1-2} g_{22}^{m_1}, \quad \alpha_2(g) = |g_{22}|^{-m_2+i\sigma_2-2} g_{22}^{m_2},$$

and the representation operators are given by the formula

$$T_g \tilde{f}(u_1,u_2) = \frac{\alpha_1(u_1 g)}{\alpha_1(\overline{u_1 g})}\frac{\alpha_2(u_2 g)}{\alpha_2(\overline{u_2 g})}\tilde{f}(u_1\overline{g}, u_2\overline{g}).$$

The transition to this realization is given by the formula

$$\tilde{f}(u_1,u_2) = \pi\alpha_1(u_1)\alpha_2(u_2) f(z_1,z_2) \quad \text{for } u_1 = \delta_1\zeta_1 z_1,\ u_2 = \delta_2\zeta_2 z_2.$$

Put

$$\tilde{f}(u,\chi) = \sqrt{\pi\alpha(u)}\, f(z,\chi) \quad \text{for } u = \delta\zeta.$$

The correspondence $f(z,\chi) \to \tilde{f}(u,\chi)$ thus established is an isometric mapping of the space $\mathfrak{M}$ onto the Hilbert space $\tilde{\mathfrak{M}}$ of all measurable functions $\tilde{f}(u,\chi)$, $\chi \in X_0^+$, satisfying the conditions

$$\tilde{f}(\gamma u,\chi) = \alpha(\gamma)\tilde{f}(u,\chi), \quad \int_{X_0^+} \omega(\chi) d\chi \int |\tilde{f}(u,\chi)|^2 du < \infty$$

with scalar product

$$(\tilde{f}_1,\tilde{f}_2) = 2\int_{X_0^+} \omega(\chi) d\chi \int \tilde{f}_1(u,\chi)\overline{\tilde{f}_2(u,\chi)}\, du.$$

The isometric mapping S then passes into the isometric mapping $\tilde{S}$, defined by the formula

$$\tilde{f}(u,\chi) = \iint \tilde{f}(u_1,u_2) b(u_1,u_2,u,\chi)\, du_1 du_2, \tag{6'}$$

where

$$\begin{aligned} b(u_1,u_2,u,\chi) = \pi^{3/2} &|\xi_2\eta_1 - \xi_1\eta_2|^{-\frac{m+m_1+m_2}{2} - i\frac{\sigma+\sigma_1+\sigma_2}{2} - 1} (\xi_2\eta_1 - \xi_1\eta_2)^{-\frac{m+m_1+m_2}{2}} \\ &\times |\xi\eta_1 - \xi_1\eta|^{-\frac{m-m_1+m_2}{2} + i\frac{\sigma-\sigma_1+\sigma_2}{2} - 1} (\xi\eta_1 - \xi_1\eta)^{\frac{m-m_1+m_2}{2}} \\ &\times |\xi_2\eta - \xi\eta_2|^{-\frac{m+m_1-m_2}{2} + i\frac{\sigma+\sigma_1-\sigma_2}{2} - 1} (\xi_2\eta - \xi\eta_2)^{\frac{m+m_1-m_2}{2}}, \end{aligned}$$

for

$$u = \begin{pmatrix} \bar{\eta} & -\bar{\xi} \\ \xi & \eta \end{pmatrix}, \quad u_1 = \begin{pmatrix} \bar{\eta}_1 & -\bar{\xi}_1 \\ \xi_1 & \eta_1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \bar{\eta}_2 & -\bar{\xi}_2 \\ \xi_2 & \eta_2 \end{pmatrix}.$$

The assertion of the preceding theorem remains valid if one replaces $f(z_1,z_2)$, $f(z,\chi)$, $L^2(Z \times Z)$, $\mathfrak{M}$, and formula (6), respectively, by $\tilde{f}(u_1,u_2)$, $\tilde{f}(u,\chi)$, $L^2_{m_1}(\mathfrak{U}) \times L^2_{m_2}(\mathfrak{U})$, $\tilde{\mathfrak{M}}$, and formula (6′).

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