MATHEMATICS

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1958

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Abstract

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ANALYTIC CAPACITY OF SETS

AND SOME OF ITS PROPERTIES

(Presented by Academician A. N. Kolmogorov, 8 X 1958)

The concept of the analytic capacity of a set arose in connection with the study of sets on which every continuous function can be expanded in a uniformly convergent series of rational functions. The results pertaining to this problem will be published later; here only some preliminary propositions will be formulated.

§ 1. **Definition of analytic capacity.** Let e be some bounded set lying in the complex plane τ; let $\varphi(z)$ be a complex function of the complex variable $z = x + iy$, possessing the following properties: a) the function $\varphi(z)$ is defined everywhere on $\tau - e$, and $|\varphi(z)| \leqslant 1$; b) $\varphi(z)$ is analytic everywhere outside the set e, i.e., for every point of $\tau - e$ there exists a neighborhood in which $\varphi(z)$ is expandable in a uniformly convergent, throughout the whole neighborhood, series of polynomials; c) $\lim\limits_{z\to\infty} \varphi(z) = 0$.

Put $\gamma(e, \varphi) = \lim\limits_{z\to\infty} |z\varphi(z)|$ and $\gamma(e) = \sup\limits_{\varphi} \gamma(e, \varphi)$ (the least upper bound is taken over all functions $\varphi(z)$ satisfying conditions a), b), c)).

The felicitous name for the quantity $\gamma(e)$ was devised by V. D. Erokhin—he proposed calling the quantity $\gamma(e)$ the **analytic capacity** of the set e.

As an explanation of item b), we note that for every $\varepsilon > 0$ one can specify a closed set $e_\varepsilon \subset e$ and a function $\varphi_e^\varepsilon(z)$, satisfying conditions a), b), c) (with respect to e_ε), such that $\gamma(e_\varepsilon) \geqslant \gamma(e) - \varepsilon$ and $\lim\limits_{z\to\infty} z\varphi_e^\varepsilon(z) = \gamma_\varepsilon \geqslant \gamma(e) - \varepsilon$, where γ_ε is a real number. Put $\varphi_e(z) = \lim\limits_{\varepsilon\to 0} \varphi_e^\varepsilon(z)$. We shall call this function $\varphi_e(z)$, for the set e, the **Ahlfors function**.

Some of the simplest properties of $\gamma(e)$ and φ_e:

1. For every set e, $\gamma(e) \geqslant 0$.
2. If $e'' \supset e'$, then $\gamma(e'') \geqslant \gamma(e')$.
3. If the sets e' and e'' are congruent, then $\gamma(e') = \gamma(e'')$.
4. For every closed set e, $\gamma(e) = \gamma(\tilde{e})$, where $\tilde{e}$ is the boundary of the set e.

5. The function $\gamma(e)$ has dimension 1, i.e., under a dilation of the set by a factor of k, its analytic capacity increases by a factor of k.

6. For every set e,
$$\gamma(e) = \frac{1}{2\pi} h_1(e),$$
where $h_1(e)$ is the Hausdorff length of the set e.

7. For every set e and $\varepsilon > 0$, one can specify a closed (in the plane) set $e' \subset e$ such that $\gamma(e') \geqslant \gamma(e) - \varepsilon$.

8. If the set e is closed, then $\varphi_e(z)$ satisfies conditions a), b), c), and every function $\varphi(z)$ satisfying conditions a), b), c) (with respect to e), for which $\lim\limits_{z\to\infty} z\varphi(z) = \gamma(e)$, is identically equal (outside the set e) to the function $\varphi_e(z)$.

9. If the set e consists of n closed components distinct from a point, then, by Ahlfors' theorem, the function $\varphi_e(z)$ assumes the value zero exactly n times (counting multiplicities of zeros).

10. If the set e consists of n closed components distinct from a point, then
$$\gamma(e) = C(e) \prod_{k=1}^{n-1} e^{-h_k} \leqslant C(e),$$
where $C(e)$ is the harmonic capacity of the set e, and $h_k > 0$ are the values of the Green function $g_e(\infty, z)$ at the zeros $\{z_k\}$, distinct from $z = \infty$, of the Ahlfors function $\varphi_e(z)$.

11. If e is a continuum, then $\gamma(e) = C(e)$, since $-\lg|\varphi_e(z)|$ coincides with the Green function $g_e(\infty, z)$ of the set e.

12. If e is a continuum, then $A\gamma(e) \leqslant d(e) \leqslant B\gamma(e)$, where $d(e)$ is the diameter of the set e, and $A > 0$ and $B > 0$ are absolute constants.

13. If the planar measure of the set e is equal to s, then $\gamma(e) \geqslant \sqrt{s}/2\sqrt{\pi}$.

§ 2. Estimation, in terms of $\gamma(e)$, of the coefficients of the expansion of an analytic function.

Lemma 1. Let g be some domain; let e be a subset of this domain at a positive distance from its boundary. Then every function $f(z)$, analytic in $g - e$ and bounded by the constant m, can be represented in the form $f(z) = C + \varphi(z) + g(z)$, where C is a constant; $\varphi(z)$ is a function analytic outside the set e, uniformly bounded (in modulus) by the constant βm and equal to zero at $z = \infty$; $g(z)$ is a function analytic in the domain g, also bounded by the constant βm ($\beta > 0$ is a constant depending only on g and e).

Let us note that if $g - e$ contains an annulus bounded by two concentric circles with ratio of radii $\rho < 1$, then

$$\beta \leqslant \frac{2}{1-\rho}.$$

Lemma 2. Every set e can be covered by a closed disk g_e with radius $r_e \leqslant r_0 d(e)$, where $r_0 < 1$ is an absolute constant, and $d(e)$ is the one-dimensional diameter of the set e.

Theorem 1. If the function $\varphi(z)$ satisfies conditions a), b), c), then outside the disk of minimal radius containing the set e, it is representable in the form of the series

$$\varphi(z) = \sum_{k=1}^{\infty} \frac{c_k}{(z-a)^k},$$

where a is the center of the mentioned disk, and the coefficients c_k satisfy the inequality

$$|c_k| \leqslant A\gamma(e)[d(e)]^{k-1}$$

($A > 0$ is an absolute constant).

Proof. Consider the annulus bounded by the concentric circles $K_1\{|z-a| = r_e\}$ and $K_2\{|z-a| = \frac{1}{2}[d(e)+r_e]\}$. For $|z-a| \geqslant d(e)$ we have

$$\begin{aligned} |f(z)| &= \frac{1}{2\pi}\left|\int_{K_2} \frac{f(\xi)}{\xi - z}\, d\xi\right| = \frac{1}{2\pi}\left|\int_{K_2} [c_z + \varphi_z(\xi) + g_z(\xi)]\, d\xi\right| \\ &= \frac{1}{2\pi}\left|\int_{K_2} \varphi_z(\xi)\, d\xi\right| \leqslant \frac{1}{2\pi} \max_{K_2} |\varphi_z(\xi)| \left|\int_{K_2} \frac{\varphi_z(\xi)\, d\xi}{\max|\varphi_z(\xi)|}\right| \\ &\leqslant \max|\varphi_z(\xi)|\, \gamma(e) \leqslant \beta m_z \gamma(e) \leqslant A\frac{\gamma(e)}{d(e)}. \end{aligned}$$

Consequently,

$$|c_k| = \frac{1}{2\pi}\left|\int_{|z-a|=d(e)} f(z)(z-a)^{k-1}\, dz\right| \leqslant A\gamma(e)[d(e)]^{k-1}.$$

Corollary 1. In order that every bounded function analytic outside the set e be constant, it is necessary and sufficient that the analytic capacity of the set e be equal to zero.

Corollary 2. Let g be some domain with rectifiable boundary γ, and let $e \subset g$ be a set at a positive distance from γ. Then for every function $f(z)$ analytic in $g - e$ the inequality

$$\left|\int_{\gamma} f(z)\, dz\right| \leq B \max_{g-e} |f(z)|\, \gamma(e),$$

holds, where $B > 0$ is an absolute constant.

§ 3. **Condensation points of a set.** Some notation: Ce is the complement of the set e in the whole plane τ; σ_z^r is the open disk of radius r with center at the point z; K_z^n is the closed annulus bounded by two circles with radii $r_n = 1/2^n$ and $r_{n+1} = 1/2^{n+1}$ with common center at the point z; $P_z^r(e) = \dfrac{1}{r}\gamma(e \cap \sigma_z^r)$ is the mean analytic density of the set e in the disk σ_z^r; $P_z(e) = \inf_{r\to 0} P_z^r(e)$ is the analytic density of the set e at the point z; e_∞ is the set of all such points z of the plane τ for each of which

$$\sup_{r\to 0} \frac{1}{r^2}\gamma(Ce \cap \sigma_z^r) = \infty.$$

Definition. A point $z \in \tau$ will be called a **condensation point** of the set e if

$$\sum_{n=1}^{\infty} \left(\frac{1}{r_n}\right)^2 \gamma(Ce \cap K_z^n) < \infty.$$

If the indicated series diverges, then the point z will be called a **rarefaction point** of the set e.

Lemma 3. *If the set e is a simply connected domain and $d(e) < 1$, then the quantity*

$$r_z(e) = \left[\sum_{n=1}^{\infty} 4^n \gamma(Ce \cap K_z^n)\right]^{-1}$$

is comparable with the distance $\rho(z, Ce)$ from the point z to the set Ce, i.e. there exist positive absolute constants C', C'' such that

$$C'\rho(z, Ce) \le r_z(e) \le C''\rho(z, Ce).$$

From this lemma, in particular, it follows that every interior point of the set e is its condensation point.

Lemma 4. *For every set e and disk σ_z^r the inequality*

$$P_z^r(e) \ge C_1 \frac{s}{r^2},$$

holds, where s is the exterior planar measure of the set $e_\infty \cap \sigma_z^r$, and $C_1 > 0$ is an absolute constant.

Proof. Cover the set $e_\infty \cap \sigma_z^r$ by a system of disks $\{\sigma_{z_k}^{r_k}\}$ such that

$$\gamma(Ce \cap \sigma_{z_k}^{r_k}) \geq \frac{1}{r}(r_k)^2.$$

From this system choose a finite subsystem of disks $\sigma_k = \sigma_{z_k}^{r_k}$ $(k = 1, 2, \dots, n)$ such that

$$s \geq \sum_{k=1}^{n} \pi(r_k)^2 \geq C_2 s$$

($C_2 > 0$ is some sufficiently small, but absolute constant) and such that the disks $\sigma'_k = \sigma_{z_k}^{4r_k}$ $(k = 1, 2, \dots, n)$ are pairwise disjoint. For each of the sets $\{Ce \cap \sigma_k\}$ fix a function $\varphi_k(\xi)$, analytic outside the set $Ce \cap \sigma_k$, bounded (in modulus) by the constant

$$\frac{(r_k)^2}{r\,\gamma(Ce \cap \sigma_k)}$$

and such that

$$\lim_{\xi \to \infty} \xi \varphi_k(\xi) = \frac{1}{2r}(r_k)^2.$$

Consider the function

$$\varphi(\xi) = \frac{1}{1+m} \sum_{k=1}^{n} \varphi_k(\xi),$$

where m is the maximum of $\left|\sum_{k=1}^{n} \varphi_k(\xi)\right|$. For the function $\varphi(\xi)$ the corresponding quantity is

$$\gamma(Ce \cap \sigma_z^r, \varphi) = \lim_{\xi \to \infty} \xi \varphi(\xi) = \frac{1}{1+m} \sum_{k=1}^{n} \gamma(Ce \cap \sigma_k, \varphi_k) \geq$$

$$\geq \frac{1}{1+m} \sum_{k=1}^{n} \frac{1}{r}(r_k)^2 \geq \frac{C_2 S}{\pi r(1+m)}.$$

Assuming that ξ does not belong to any of the disks $\{\sigma'_k\}$, from Theorem 1 we obtain

$$\left|\sum_{k=1}^{n} \varphi_k(\xi)\right| \leq \sum_{k=1}^{n} |\varphi_k(\xi)| = \sum_{k=1}^{n} \left|\sum_{q=1}^{\infty} \frac{C_q^k}{(\xi - z_k)^2}\right| \leq$$

$$\leq \sum_{k=1}^{n} \sum_{q=1}^{\infty} \max |\varphi_k(\xi)| \frac{A\gamma(Ce \cap \sigma_k)[d(Ce \cap \sigma_k)]^{q-1}}{|\xi - z_k|^q} \leq$$

$$\leq \frac{C_3}{r} 2\sqrt{\pi S} \leq \frac{2C_3}{r} \sqrt{\pi r^2} = C_4,$$

where C_4 is an absolute constant.

If, however, ξ belongs to some one of the disks $\{\sigma_k'\}$, then

$$\left| \sum_{k=1}^{n} \varphi_k(\xi) \right| \leq 1 + C_4.$$

Thus, $m \leq C_4 + 1$. Consequently,

$$\gamma(Ce \cap \sigma_z^r) \geq \gamma(Ce \cap \sigma_z^r, \varphi) \geq \frac{C_5 S}{r},$$

i.e.

$$P_z^r(e) \geq \frac{C_1 S}{r^2}.$$

The lemma is proved.

Lemma 5. For an arbitrary set e, at almost every one of its points of rarefaction (up to a set of planar measure 0), the analytic density of the complement is positive.

We omit the proof of Lemma 5 because of its cumbersomeness.

Theorem 2. If the set e has no points of condensation, then for all r and z

$$P_z^r(Ce) \geq \pi C_1$$

(see Lemma 2).

It will be proved below that, under the assumptions of Theorem 2, $P_z^r(Ce)$ turns out to be identically equal to 1.

Received
30 IX 1958

Note: Figure translations are in progress. See original paper for figures.