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**Abstract**

**Full Text**

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## ON THE PROPAGATION OF WAVES IN A LOCALLY ISOTROPIC TURBULENT MEDIUM WITH SMOOTHLY VARYING CHARACTERISTICS

*(Presented by Academician A. N. Kolmogorov, January 6, 1958)*

In works <sup>(1,2)</sup>, using the methods developed in <sup>(3,4)</sup>, the problem was considered of fluctuations of the amplitude and phase of short waves ( $\lambda \ll l_0$ , where  $\lambda$  is the wavelength,  $l_0$  is the inner scale of turbulence) propagating in a locally isotropic turbulent medium whose local statistical properties are the same along the entire path of wave propagation. However, in calculating fluctuations of the amplitude and phase of waves propagating in the real atmosphere (or in the sea), it is necessary to take into account that, in different sections along the “ray” path, the turbulence may have different intensity (for example, the “intensity” of atmospheric turbulence depends strongly on height).

We shall assume that in regions small in comparison with the outer scale of turbulence  $L_0$ , the turbulence is locally isotropic, i.e., that the structure functions  $[f(\mathbf{r}_1) - f(\mathbf{r}_2)]^2 = D_f(\mathbf{r}_1, \mathbf{r}_2)$  depend only on  $|\mathbf{r}_1 - \mathbf{r}_2|$ . However, when the pair of points  $\mathbf{r}_1, \mathbf{r}_2$  is displaced by a distance of order  $L_0$ ,  $D_f(\mathbf{r}_1, \mathbf{r}_2)$  may change by some factor depending on  $\frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2)$ , i.e., on the position of the center of the segment  $\mathbf{r}_1 - \mathbf{r}_2$ . We shall suppose that  $D_f(\mathbf{r}_1, \mathbf{r}_2)$  has the form

$$D_f(\mathbf{r}_1, \mathbf{r}_2) = C_f^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) D_0(|\mathbf{r}_1 - \mathbf{r}_2|). \quad (1)$$

The function  $C_f^2(\mathbf{r})$  changes appreciably only when  $\mathbf{r}$  changes by an amount of order  $L_0$ .

In the case when  $C_f^2 = \text{const}$ , the structure function  $D_f(\mathbf{r}_1, \mathbf{r}_2)$  can be represented by the spectral expansion <sup>(5)</sup>

$$D_f(\mathbf{r}_1, \mathbf{r}_2) = 2 \iiint_{-\infty}^{\infty} [1 - \cos \chi(\mathbf{r}_1 - \mathbf{r}_2)] \Phi_f(\chi) d\chi, \quad (2)$$

where  $\Phi_f(\chi)$  is the spectral density corresponding to the function  $D_f(\mathbf{r}_1, \mathbf{r}_2)$ . In the case of variable  $C_f^2$ , one may define the expansion

$$D_f(\mathbf{r}_1, \mathbf{r}_2) = 2C_f^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) \iiint_{-\infty}^{\infty} [1 - \cos \chi(\mathbf{r}_1 - \mathbf{r}_2)] \Phi_0(\chi) d\chi, \quad (3)$$

where  $\Phi(\chi)$  also depends on the position of the center of the pair of points  $\mathbf{r}_1, \mathbf{r}_2$ :

$$\Phi_f(\chi) = C_f^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) \Phi_0(\chi). \quad (4)$$

As is known <sup>(6,7)</sup>, the structure function of the concentration of a conservative passive admixture in a turbulent flow is expressed by the “two-thirds law.” For the refractive index of the atmosphere  $n$ , which with some-

under the same assumptions can also be considered as a conservative passive admixture; this law is given by the formula

$$\overline{[n(\mathbf{r}_1) - n(\mathbf{r}_2)]^2} = D_n(\mathbf{r}_1, \mathbf{r}_2) = C_n^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) |\mathbf{r}_1 - \mathbf{r}_2|^{2/3}, \quad (l_0 \ll |\mathbf{r}_1 - \mathbf{r}_2| \ll L_0). \quad (5)$$

To the structure function (5) there corresponds the spectral density <sup>(2)</sup>

$$\Phi_n(\varkappa) = 0.033 C_n^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) \varkappa^{-11/3}. \quad (6)$$

In the region  $|\mathbf{r}_1 - \mathbf{r}_2| \ll l_0$ ,  $D_n(\mathbf{r}_1, \mathbf{r}_2)$ , as is known <sup>(6)</sup>, has a quadratic character:

$$D_n(\mathbf{r}_1, \mathbf{r}_2) = C_n^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) l_0^{2/3} \left( \frac{|\mathbf{r}_1 - \mathbf{r}_2|}{l_0} \right)^2. \quad (7)$$

Such a form of the structure function in the region  $|\mathbf{r}_1 - \mathbf{r}_2| \ll l_0$  corresponds to a rapid decrease to zero of  $\Phi_n(\varkappa)$  in the region  $\varkappa \sim \varkappa_m \approx 2\pi/l_0$ . Since the exact form of the spectral density  $\Phi_n(\varkappa)$  in the region  $\varkappa \sim \varkappa_m$  is immaterial for us, we shall simply put

$$\Phi_n(\varkappa) = \begin{cases} 0.033 C_n^2 \left( \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \right) \varkappa^{-11/3}, & \text{for } \varkappa < \varkappa_m, \\ 0, & \text{for } \varkappa > \varkappa_m; \end{cases} \quad (8)$$

$l_0$  and  $\varkappa_m$  are connected by the relation  $\varkappa_m l_0 = 5.48$  <sup>(2)</sup>.

In solving the problem of fluctuations of the parameters of a wave propagating in a medium with random inhomogeneities of the refractive index  $n_1(\mathbf{r}) = n(\mathbf{r}) - \bar{n}(\mathbf{r})$ , we shall start from the equation <sup>(4,3,1)</sup>

$$\frac{\partial^2 \Psi_1}{\partial y^2} + \frac{\partial^2 \Psi_1}{\partial z^2} + 2ik \frac{\partial \Psi_1}{\partial x} + 2k^2 n_1(x, y, z) = 0 \quad (9)$$

(the  $x$ -axis is directed along the incident wave), which describes perturbations of the logarithm of the amplitude  $\ln(A/A_0) = \text{Re} \Psi_1$  and of the phase  $S_1 = S - S_0 = \text{Im} \Psi_1$  of a plane monochromatic wave  $A_0 e^{ikx}$  propagating along the  $x$ -axis. Equation (9) is valid under the conditions:  $\lambda \ll l_0$ ,  $\lambda^3 L \ll l_0^4$ ,  $\lambda |\nabla \Psi_1| \ll 2\pi$ ; here  $L$  is the distance traversed by the wave.

Fluctuations of the refractive index  $n_1$  may be represented in the form of a stochastic Fourier-Stieltjes integral

$$n_1(x, y, z) = n_1(x, 0, 0) + \int_{-\infty}^{\infty} \int [1 - e^{i(\varkappa_2 y + \varkappa_3 z)}] d\nu(\varkappa_2, \varkappa_3, x). \quad (10)$$

In the same form one should also seek the function  $\Psi_1(x, y, z)$ :

$$\Psi_1(x, y, z) = \Psi_1(x, 0, 0) + \int_{-\infty}^{\infty} \int [1 - e^{i(\varkappa_2 y + \varkappa_3 z)}] d\psi(\varkappa_2, \varkappa_3, x). \quad (11)$$

Substituting these expansions into equation (9), we obtain an ordinary differential equation, from which the function is determined

$$d\psi(\varkappa_2, \varkappa_3, L) = ik \int_0^L [e^{i\varkappa^2(L-x)/(2k)} d_x \nu(\varkappa_2, \varkappa_3, x)] dx. \quad (12)$$

On the basis of (12) one can express the spectral densities of the fluctuations of the amplitude and phase of the wave in the plane  $x = L$  through the spectral density of the fluctuations of the refractive index. After some transformations, in the course of which the condition  $\lambda \ll l_0$  is used (see the analogous derivation in <sup>(2)</sup>), one may obtain the formulas

$$F_A(\varkappa) = 2\pi k^2 \Phi_0(\varkappa) \int_0^L C_n^2(r) \sin^2 \frac{\varkappa^2(L-x)}{2k} dx; \quad (13)$$

$$F_S(\varkappa) = 2\pi k^2 \Phi_0(\varkappa) \int_0^L C_n^2(\mathbf{r}) \cos^2 \frac{\varkappa^2(L-x)}{2k} dx, \quad (14)$$

which express the two-dimensional spectral densities of the structural (or correlation) functions of the amplitude fluctuations  $F_A(\varkappa)$  and phase fluctuations  $F_S(\varkappa)$  in terms of the three-dimensional spectral density  $\Phi_n(\varkappa) = C_n^2(\mathbf{r}) \Phi_0(\varkappa)$  of fluctuations of the refractive index.

With the aid of (13) and (14), one can write the correlation function of fluctuations of the logarithm of the wave amplitude in the plane  $x = L$  as the two-dimensional Fourier transform of the function  $F_A(\nu)$  in the plane  $x = L$ :

$$B_A(\rho) = 4\pi^2 k^2 \int_0^L C_n^2(\mathbf{r}) dx \int_0^\infty J_0(\nu\rho) \Phi_0(\nu) \sin^2 \frac{\nu^2(L-x)}{2k} \nu d\nu. \quad (15)$$

The mean square of the fluctuations of the logarithm of the amplitude is  $\overline{\chi^2} = \overline{(\ln A/A_0)^2} = B_A(0)$ :

$$\overline{\chi^2} = 4\pi^2 k^2 \int_0^L C_n^2(\mathbf{r}) dx \int_0^\infty \Phi_0(\nu) \sin^2 \frac{\nu^2(L-x)}{2k} \nu d\nu. \quad (16)$$

For the structural function of the wave phase we obtain from (14):

$$D_S(\rho) = \overline{(S_1 - S_1')^2} = 8\pi^2 k^2 \int_0^L C_n^2(\mathbf{r}) dx \int_0^\infty [1 - J_0(\nu\rho)] \Phi_0(\nu) \cos^2 \frac{\nu^2(L-x)}{2k} \nu d\nu. \quad (17)$$

Formulas (15), (16), and (17) solve the problem posed.

Let us consider the practically important case when fluctuations of the refractive index are described by the "two-thirds law." In this case  $\Phi_n(\nu)$  is determined by formula (8). For the quantity  $\overline{\chi^2}$  we obtain the formula

$$\overline{\chi^2} = 1.30k^2 \int_0^L C_n^2(\mathbf{r}) dx \int_0^{\nu_m} \nu^{-11/3} \sin^2 \frac{\nu^2(L-x)}{2k} \nu d\nu. \quad (18)$$

For the case when  $\nu_m^2 L \ll k$ , or  $\sqrt{\lambda L} \ll l_0$  (i.e., in the geometrical-optics region), replacing the sine by its argument, we obtain

$$\overline{\chi^2} = 7.37l_0^{-7/3} \int_0^L C_n^2(\mathbf{r})(L-x)^2 dx. \quad (19)$$

For the case when  $l_0 \ll \sqrt{\lambda L} \ll L_0$ , extending the integration in (18) to infinity, we obtain:

$$\overline{\chi^2} = 0.56k^{7/6} \int_0^L C_n^2(\mathbf{r})(L-x)^{5/6} dx. \quad (20)$$

Investigating the general expression (15) for  $B_A(\rho)$ , one can verify that, in the case when  $\sqrt{\lambda L} \ll l_0$ , the correlation scale of the wave-amplitude fluctuations

is of order  $l_0$ , while in the case when  $l_0 \ll \sqrt{\lambda L} \ll L_0$ , it coincides in order of magnitude with  $\sqrt{\lambda L}$ . When the form of the function  $C_n^2(\mathbf{r})$  changes, the correlation scale of the fluctuations changes only slightly.

For the structural function of the wave-phase fluctuations in the case when  $\Phi_n(\mathcal{N})$  is specified by formula (8), one can obtain the following expressions:

a) The case  $\sqrt{\lambda L} \ll l_0$ :

$$D_S(\rho) = 3.44 k^2 l_0^{-1/3} \rho^2 \int_0^L C_n^2(\mathbf{r}) dx \quad (\text{for } \rho \ll l_0); \quad (21)$$

$$D_S(\rho) = 2.91 k^2 \rho^{5/3} \int_0^L C_n^2(\mathbf{r}) dx \quad (\text{for } \rho \gg l_0). \quad (22)$$

b) The case  $l_0 \ll \sqrt{\lambda L} \ll L_0$ :

$$D_S(\rho) = 1.72 k^2 l_0^{-1/3} \rho^2 \int_0^L C_n^2(\mathbf{r}) dx \quad (\rho \ll l_0); \quad (23)$$

$$D_S(\rho) = 1.46 k^2 \rho^{5/3} \int_0^L C_n^2(\mathbf{r}) dx \quad (l_0 \ll \rho \ll \sqrt{\lambda L}); \quad (24)$$

$$D_S(\rho) = 2.91 k^2 \rho^{5/3} \int_0^L C_n^2(\mathbf{r}) dx \quad (\sqrt{\lambda L} \ll \rho \ll L_0). \quad (25)$$

(Formula (25) is practically valid already for  $\rho \approx \sqrt{\lambda L}$ .)

The contribution of different inhomogeneities to fluctuations of the wave phase does not depend on the coordinates of these inhomogeneities, whereas their contribution to fluctuations of the wave amplitude increases with their distance from the point of observation.

Formulas (20) and (25) can be used to explain certain features of the scintillation of stars. In a coordinate system with its center at the point of observation,

$$\overline{\chi^2} = 0.56 k^{7/6} \int_0^L C_n^2(z) z^{5/6} dz \quad (l_0 \ll \sqrt{\lambda L} \ll L_0); \quad (26)$$

$$D_S(\rho) = 2.91 k^2 \rho^{5/3} \int_0^L C_n^2(z) dz \quad (\rho \ll L_0). \quad (27)$$

The integrals in (26) and (27) may be extended to infinity (in this case, in the condition  $l_0 \ll \sqrt{\lambda L} \ll L_0$ ,  $L$  should be understood as a quantity of the order

of the thickness of the atmospheric layer in which appreciable fluctuations of the refractive index exist).

The quantity  $C_n^2(z)$  in the atmosphere is maximal near the Earth's surface (in a layer of the order of several hundred meters)<sup>2</sup>, and then decreases rather rapidly with height. Therefore, in the integral (27) the main contribution is made by the lower layers of the atmosphere. At the same time, because of the presence of the factor  $z^{5/6}$  in the integral (26), the weight of the lower layers is considerably weakened. Thus, the lower layers of the atmosphere exert the principal influence on the "jitter" of stellar images in telescopes, whereas higher layers are more important for stellar scintillation. This conclusion is in good agreement with astronomical observations<sup>8,9</sup>.

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## REFERENCES

- <sup>1</sup> V. I. Tatarskii, *Dokl. Akad. Nauk SSSR*, **107**, No. 2, 245 (1956).
- <sup>2</sup> V. I. Tatarskii, Dissertation, Acoustics Institute, Academy of Sciences of the USSR, Moscow, 1956.
- <sup>3</sup> A. M. Obukhov, *Izv. Akad. Nauk SSSR, Ser. Geofiz.*, No. 2 (1953).
- <sup>4</sup> S. M. Rytov, *Izv. Akad. Nauk SSSR, Ser. Fiz.*, No. 2 (1937).
- <sup>5</sup> A. M. Yaglom, *Theory of Probability and Its Applications*, **2**, issue 3 (1957).
- <sup>6</sup> A. M. Obukhov, *Izv. Akad. Nauk SSSR, Ser. Geogr. i Geofiz.*, No. 1 (1949).
- <sup>7</sup> A. M. Yaglom, *Dokl. Akad. Nauk SSSR*, **69**, No. 6 (1949).
- <sup>8</sup> F. Gifford, A. H. Mikesell, *Weather*, **8**, 195 (1953).
- <sup>9</sup> F. Gifford, H. Johnson, A. Wilson, *Astr. J.*, **60**, No. 5 (1955).

*Note: Figure translations are in progress. See original paper for figures.*

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