# MATHEMATICS

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## Abstract

## Full Text

*MATHEMATICS*

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# THE CAUCHY PROBLEM FOR GENERAL SYSTEMS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

*(Presented by Academician I. G. Petrovskii on 17 I 1958)*

**1.** This paper studies the Cauchy problem for a system of linear partial differential equations of general form. The number of equations is assumed to be equal to the number of unknown functions.

Let $t = t_0$ be a hyperplane in the space $(t, x_1, \dots, x_n)$ on which the initial conditions are prescribed. The coefficients of the system are assumed to depend only on the variable $t$. Replace in the system all lower derivatives with respect to $t$ of the unknown functions by new unknown functions; after this the order of differentiation with respect to $t$ in the system will be no higher than the first, and the system can be written in the form

$$M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)\frac{\partial \bar{u}}{\partial t} = L_1\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)\bar{u}. \tag{1}$$

The initial conditions will be written as

$$\bar{u}(t_0, x_k) = \bar{\varphi}(x_k). \tag{2}$$

Here the bar above denotes vector functions with $N$ components;
$M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)$ and $L_1\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)$ are square matrix polynomials in the operators

$$\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n},$$

with coefficients depending on $t$. Instead of the arguments $(x_1, \dots, x_n)$ we shall briefly write $(x_k)$.

We shall seek the solution of problem (1)–(2) for $t \geq t_0$. Such a system of special form was first studied by S. L. Sobolev ($^1$). Other systems of this type were investigated by M. I. Vishik ($^2$).

If it is assumed that the coefficients of the matrix $M$ possess derivatives, then system (1) can be written in the form

$$\frac{\partial}{\partial t}\left[M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)\bar{u}\right] = L\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right)\bar{u}. \tag{1'}$$

We shall assume the coefficients in system $(1')$ to be continuous functions of $t$ for $t_0 \le t \le T$. After the Fourier transform

$$v_i = \frac{1}{(2\pi)^n}\int u_i e^{-i(\alpha,x)}\,dx_1 \cdots dx_n,$$

where $u_i$ and $v_i$ are components of the vectors $\bar{u}$ and $\bar{v}$;

$$(\alpha, x) = \sum_{k=1}^{n} \alpha_k x_k,$$

we obtain

$$\frac{d}{dt}\left[M(t, \alpha_k)\bar{v}\right] = L(t, \alpha_k)\bar{v}. \tag{3}$$

Let the vector functions

$$\bar{v}^{\,l} = (v_1^l, \dots, v_N^l), \quad l = 1, \dots, N,$$

form a fundamental system of solutions of system (3), and let the solution matrix satisfies the condition:

$$\left\|v_i^l\right\|_{t=t_0} = E, \tag{4}$$

where $E$ is the identity matrix.

I. G. Petrovsky, for systems with matrix $M(t, \alpha_k)$ identically equal to the identity, proved (see [3], p. 3 (condition A)) that the growth of the elements of the matrix $\|v_i^l\|$ as $\alpha^2 = \sum_{k=1}^n \alpha_k^2 \to \infty$ no faster than some power of $\alpha$ is a necessary and sufficient condition for the correctness (uniformly with respect to $t_0$) of the formulation of the problem in the class of bounded functions. In the case of systems of the form (1), unlike the systems considered by I. G. Petrovsky, there may occur solutions $v_i^l$ that tend to infinity as $\alpha_k \to \alpha_k^0$, $k = 1, \dots, n$, and as $t \to t_0 + \tau$. For example, for the equation $\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right) = -tu$ we have $v = e^{t/2\alpha^2}$, and $v \to \infty$ when $t \neq 0$ and $\alpha \to 0$.

Solving the Cauchy problem for arbitrary initial conditions, even if as smooth or analytic as desired and tending to zero arbitrarily rapidly at infinity on the real axis, does not make it possible to represent the solution by means of the Fourier integral. Consequently, for arbitrary initial functions of the indicated classes there does not always exist an absolutely integrable solution. However, if one takes such initial functions whose Fourier transform decreases as $\alpha \to 0$ like $e^{-T/|\alpha|^2}$, then the solution of this class of initial functions can be represented by a Fourier integral.

We note that in the case of unbounded growth of some element of the matrix $\|v_i^l\|$ as $t \to t_0 + \tau,\ \tau > 0$, and $\alpha_k \to \alpha_k^0$, the Cauchy problem for such a system is incorrectly posed in the class of bounded functions.

2. **Theorem 1.** *If the matrix $\|v_i^l\|$ remains bounded in any finite part of the space $(\alpha_1, \dots, \alpha_n)$ for $t_0 \leqslant t \leqslant T$ and grows no faster than $\alpha^p,\ p > 0$, as $\alpha \to \infty$, then the formulas*

$$u_i = \int \sum_{l=1}^{N} v_i^l E_{\varphi_l} e^{i(\alpha,x)}\, dx_1 \dots dx_n, \tag{5}$$

*where $E_{\varphi_l}$ is the Fourier transform of the function $\varphi_l$, give a solution of the Cauchy problem $(1')$–(2) provided the following condition is fulfilled:*

*The initial functions $\varphi_i$ possess absolutely integrable derivatives, with the square, up to order $\lambda = [n/2] + k + p + 1$, where $k$ is the highest order of the derivatives in the operators $M$ and $L$.*

If we take $\lambda = [n/2] + p + 1$, then we obtain a generalized solution which is the uniform limit of solutions.

**Theorem 2.** *If the terms of the matrix $\|v_i^l\|$ satisfy, for $\alpha \leqslant 1$ and for $t_0 \leqslant t \leqslant T$, the condition*:

$$|v_i^l| \leqslant \frac{A_1}{\alpha^q}, \qquad q > 0, \tag{$A_1$}$$

*and, for $\alpha > 1,\ t_0 \leqslant t \leqslant T$, the condition:*

$$|v_i^l| \leqslant A_2 \alpha^p, \qquad p > 0, \tag{$A_2$}$$

*then formula (5) gives a solution of the problem $(1')$–(2), if the initial functions $\varphi_i$ satisfy the following requirements:*

a$_1$) all moments of $\varphi_i$ up to order $q_0 = q - n + 1$ are equal to zero, i.e.

$$\int \varphi_i (x_1)^{s_1} \dots (x_n)^{s_n} dx_1 \dots dx_n = 0 \tag{6}$$

for all integers $s_k \geqslant 0$ such that $s_1 + s_2 + \dots + s_n \leqslant q_0$;

* Here and below $A_k$ denote positive constants.

$a_2$) the functions $\varphi_i$ have square-integrable derivatives up to order $\lambda = [n/2] + p + k + 1$, where $k$ is the order of the highest derivatives in the operators $M$ and $L$.

If one takes $\lambda = [n/2] + p + 1$, then one obtains a generalized solution which is the uniform limit of solutions.

Let us note that $v_i^l$ can tend to infinity only for those values $\alpha_k$ for which $\det M$ vanishes.

**Remark 1.** If condition $(A_1)$ is not satisfied, then under the assumption that $|\det M(t, \alpha_k)| \geq A_3 \alpha^r$ for $\alpha \leq 1$, the inequality

$$|v_i^l| \leq A_4 e^{(1/\alpha)^\sigma}, \qquad \alpha \leq 1, \tag{7}$$

always holds, where $\sigma \leq r$. Then, if conditions $(A_1)$ and $a_2$) of Theorem 2 are satisfied, the solution of problem $(1')$–$(2)$ will exist in the class of initial functions whose Fourier transform satisfies the condition

$$|E_{\varphi_i}| \leq A_5 e^{-(1/\alpha)^\sigma}, \qquad \alpha \leq 1.$$

**Remark 2.** One may first seek the derivatives with respect to $(x_1, \dots, x_n)$ of order $q_0 = q - n + 1$ of the solution of problem $(1')$–$(2)$, and then reconstruct the solution. The derivatives $D^{q_0} u_i$ satisfy the system $(1')$ and the initial conditions

$$D^{q_0} u_i \big|_{t=t_0} = D^{q_0} \varphi_i. \tag{2'}$$

The following assertion holds:
Let conditions $(A_1)$ and $(A_2)$ be satisfied; then the formulas

$$D^{q_0} u_i = \int \sum_{l=1}^{N} E_{D^{q_0}\varphi_l} \, v_i^l e^{i(\alpha x)} \, dx_1 \cdots dx_n \tag{8}$$

give a solution of problem $(1')$–$(2')$, if the initial functions $\varphi_i$ have square-integrable derivatives up to order $\lambda = \sup\{[n/2] + p + k + 1, q_0\}$.

**Remark 3.** Let condition $(A_2)$ be satisfied, and instead of condition $(A_1)$ suppose that we have the condition

$$|v_i^l| \leq \frac{A_6}{|\omega(\alpha_1, \dots, \alpha_n)|^q}, \qquad q \geq 0, \tag{$A_1^*$}$$

where $\omega(\alpha_1, \dots, \alpha_n) = 0$ is some real algebraic surface. This case may occur if $\det M(t, \alpha_k)$ vanishes when $\omega(\alpha_1, \dots, \alpha_n) = 0$. Denote by $\omega$ the operator

$$\omega = \omega\left(\frac{1}{i}\frac{\partial}{\partial x_1}, \dots, \frac{1}{i}\frac{\partial}{\partial x_n}\right).$$

Then the formulas

$$\omega^q u_i = \int \sum_{l=1}^{n} E_{\omega^q \varphi_l}\, v_i^l e^{i(\alpha x)}\, dx_1 \cdots dx_n \tag{9}$$

will give the operator $\omega$ applied to the solution of problem (1′)–(2), satisfying the initial conditions

$$\omega^q u_i\big|_{t=0} = \omega^q \varphi_i.$$

It is assumed that the functions $\varphi_i$ have square-integrable derivatives up to order

$$\lambda = \sup\{[n/2] + p + k + 1, qm\},$$

where $m$ is the degree of the polynomial $\omega(\alpha_1, \dots, \alpha_n)$.

3. In proving the uniqueness theorem, one establishes a theorem stating that, under a certain sufficient smoothness of the initial-data functions,

functions and the vanishing of all moments of these functions up to some order, the solution of problem (1′)–(2) will be integrable together with its derivatives up to the prescribed order.

**Theorem 3 (uniqueness).** Suppose that conditions $(A_1)$ and $(A_2)$ are satisfied and that $|\det M(t, \alpha_k)| \geq A_7 \alpha^r,\ r \geq 0.$ Then every solution of problem (1′)–(2) with zero initial conditions, growing as $|x| \to \infty$ no faster than some polynomial, satisfies the two systems:

$$M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right) \bar{u} = \bar{P}(t, x_1, \dots, x_n); \tag{10}$$

$$L\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right) \bar{u} = \frac{\partial}{\partial t}\bar{P}(t, x_1, \dots, x_n), \tag{11}$$

where the components of $\bar{P}(t, x_1, \dots, x_n)$ are polynomials in $(x_1, \dots, x_n)$ of a certain degree

$$p^* \leq 2\{([n/2] + 1)(q + r - 1) + r\} - n$$

with coefficients depending on the variable $t$, and $\bar{P}(0, x_1, \dots, x_n) \equiv 0$.

**Remark 1.** It is clear that if the systems (10) and (11) have a common nontrivial solution that vanishes at $t = t_0$ and belongs to the class under consideration, then uniqueness in this class does not hold.

**Remark 2.** If one considers the class of solutions tending to zero as

$$|x| = \sum_{i=1}^{n} x_i^2 \to \infty$$

together with derivatives up to order $k_0$, where $k_0$ is the order of the highest derivatives in the operator

$$M\left(t, \frac{1}{i}\frac{\partial}{\partial x_k}\right),$$

then in equations (10) and (11) one must set

$$\bar{P}(t, x_1, \dots, x_n) \equiv 0.$$

**Remark 3.** If $\det M(t, \alpha_k) \neq 0$ for no values of $\alpha_k$, then in (10) and (11) one must set

$$\bar{P}(t, x_1, \dots, x_n) \equiv 0.$$

The case where $\det M(t, \alpha_k) \neq 0$ was considered by me earlier [4].

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*Note: Figure translations are in progress. See original paper for figures.*

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