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Abstract

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G. Ts. Tumarkin and S. Ya. Khavinson

PROPERTIES OF EXTREMAL FUNCTIONS

IN EXTREMAL PROBLEMS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH A WEIGHTED METRIC

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In a number of works that have appeared recently (see, for example, [1–7]), a linear extremal problem in certain classes of analytic functions was connected in a dual way with a certain problem of best approximation in the conjugate space. For other problems in function theory the method of duality was applied earlier [8,9]. Of all the works devoted to the study of extremal problems by the indicated method of duality, the most general classes of functions and the most general problems were considered in the works of one of the authors of the present paper (see [7], where a detailed exposition of these results is given); however, even in those works the restrictions imposed on the classes considered were excessively burdensome.

In the present note we consider duality relations for classes of analytic functions under assumptions that are, in a certain sense, final and do not admit further generalization along this path. With the aid of these relations we investigate properties of extremal functions under assumptions considerably more general than those made previously. With appropriate additional restrictions, the theorems of many authors readily follow from our results.

In the present note we give a detailed list of theorems for two duality-related classes of functions and only at the end list other classes of functions for which analogous investigations were carried out by us.

Let us introduce the following classes of analytic functions in a finitely connected domain G with rectifiable boundary Γ.

Class $D(G)$**.** $f(z) \in D(G)$ if the functions subharmonic in the domain G

$$\ln^+ \left| \frac{f(z)}{M} \right|$$

have in G harmonic majorants $u^M(z)$ tending to zero as $M \to \infty$. The class

$D(G)$ is, obviously, a subclass of the class of functions of bounded type. (For properties of functions of the class $D(G)$, see our note [10].) (For the simply connected case the class $D(G)$ was introduced earlier by V. I. Smirnov [11] (see also [12].)

Class $D_\rho(G)$. Let the function $\rho(\zeta) \geq 0$, defined on Γ, satisfy the condition

$$\int_\Gamma |\ln \rho(\zeta)| \frac{\partial g(\zeta, z)}{\partial n} \, ds < \infty. \tag{1}$$

where $g(\zeta, z)$ is the Green function of the domain G. We shall say that $f(z) \in D_\rho(G)$ if $f(z) \in D(G)$ and

$$\operatorname{Vrai\,max} \left| \frac{f(\zeta)}{\rho(\zeta)} \right| < \infty.$$

Class $D^1_\rho(G)$. $f(z) \in D^1_\rho(G)$, if $f(z) \in D(G)$ and $\operatorname{Vrai\,max} \left| \frac{f(\zeta)}{\rho(\zeta)} \right| \leq 1$.

If $0 < m \leq \rho(\zeta) \leq M < \infty$, then our classes D_ρ and D^1_ρ coincide, respectively, with the classes B_ρ and B^1_ρ from the paper ([3]).

Class $E_{\rho,1}(G)$. Construct the function

$$\rho(z) = \exp \left[\frac{1}{2\pi} \int_\Gamma \ln \rho(\zeta) \frac{\partial g(\zeta, z)}{\partial n} \, ds \right].$$

We shall say that $f(z) \in E_{\rho,1}(G)$, if there exists a sequence of contours $\{\Gamma^j\} \to \Gamma$, for which

$$\lim_{j \to \infty} \int_{\Gamma^j} \left| \frac{f(z)}{\rho(z)} \right| ds < \infty.$$

Theorem 1. *Let $\rho(\zeta) \geq 0$ satisfy condition (1), and let $\omega(\zeta)$ be a complex-valued function given on Γ, for which the integral $\int_\Gamma \rho(\zeta) |\omega(\zeta)| \, ds < \infty$ exists.*

Then:

1. *The following relation (of duality) holds*

$$\sup_{f \in D^1_\rho} \left| \int_\Gamma f(\zeta) \omega(\zeta) \, d\zeta \right| = \inf_{\varphi \in E_{1/\rho,1}} \int_\Gamma \rho(\zeta) |\omega(\zeta) - \varphi(\zeta)| \, ds. \tag{2}$$

2. *There exist extremal functions $f^*(z) \in D^1_\rho(G)$ and $\varphi^*(z) \in E_{1/\rho,1}(G)$ in both parts of equality (2).*

3. *In order that* $f^*(z) \in D^1_\rho(G)$ *and* $\varphi^*(z) \in E_{1/\rho,1}(G)$ *be extremal in equality (2), it is necessary and sufficient that almost everywhere on* Γ *the equality*

$$f^*(\zeta)\,[\omega(\zeta) - \varphi^*(\zeta)]\,d\zeta = e^{i\alpha}\rho(\zeta)\,|\omega(\zeta) - \varphi^*(\zeta)|\,ds, \tag{3}$$

hold, where α *is a real constant.*

4. *The extremal function* $f^*(z) \in D^1_\rho$ *is unique up to a factor* $e^{i\alpha}$*. The extremal function* $\varphi^*(z) \in E_{1/\rho,1}$ *is unique if* $f^*(z)$ *has more than* $n-1$ *zeros in* G*; otherwise* $\varphi^*(z)$ *may fail to be unique.*

Remark. If condition (1) is not satisfied, one of the classes D_ρ or $E_{1/\rho,1}$ is, generally speaking, empty.

Theorem 1 generalizes Theorem 2.1 of Ch. II of the paper ([3]), where it was required that $\rho(\zeta) > 0$ be a continuous function on Γ.

Let us note that the most important concrete functionals for the class D_ρ, for example

$$\sum_{k=1}^{N} \gamma_k f^{(\nu_k)}(\alpha_k),$$

where $\alpha_k \in G$, γ_k are arbitrary numbers, can be written in the form

$$\int_\Gamma f(\zeta)\omega(\zeta)\,d\zeta,$$

where

$$\int_\Gamma \rho(\zeta)|\omega(\zeta)|\,|d\zeta| < \infty,$$

i.e. $\omega(\zeta)$ satisfies the conditions of Theorem 1. In this and other analogous concrete cases $\omega(\zeta)$, moreover, is the boundary value of a function $\omega(z)$ meromorphic in G with a finite principal part, and, in a sufficiently narrow strip adjacent to Γ, $\omega(z)$ belongs to the class $E_{1/\rho,1}$.

Theorem 2. *Let, under the conditions of Theorem 1, the function* $\omega(\zeta)$ *coincide almost everywhere on some arc* $\gamma \subset \Gamma$ *with the boundary values of a function* $\omega(z)$*, analytic and belonging to the class* $E_{1/\rho,1}(\widetilde{G})$ *in some domain* $\widetilde{G} \subset G$ *adjacent to* γ*. Then, if* $\widehat{G} \subset \widetilde{G}$ *is any domain sufficiently close to* Γ *and adjacent to* Γ *along some arc* γ'*,*

interior to γ, then the following representations hold for the extremal function $f^*(z)$ and the extremal difference $\omega(z) - \varphi^*(z)$:

$$f^*(z) = \exp\left[\frac{1}{2\pi}\int_\gamma \ln \rho(\zeta)\frac{\partial P(\zeta, z)}{\partial n}\, ds\right] \cdot F_1(z), \tag{4}$$

$$\omega(z) - \varphi^*(z) = \psi'(z) \exp \frac{1}{2\pi}\int_\gamma \ln\left|\frac{\omega(\zeta) - \varphi^*(\zeta)}{\psi'(\zeta)}\right| \frac{\partial P(\zeta, z)}{\partial n}\, ds \cdot F_2(z), \tag{5}$$

where $z \in \widetilde{G}$; $w = \psi(z)$ is a mapping of G onto a circular canonical domain; $P(\zeta, z)$ is the complex Green function of the domain G; $F_1(z)$ and $F_2(z)$ are functions analytic in $\widetilde{G}$, continuous up to γ, and having on γ moduli equal to 1: $|F_i(\zeta)| = 1,\ \zeta \in \gamma,\ i = 1, 2$. These local representations, remarkable in that they contain no factors with singular components, as a rule do not hold if the additional restriction imposed in Theorem 2 is not imposed on $\omega(\zeta)$.

Remark. If γ coincides with one of the boundary contours γ_i, then the representations (4) and (5) hold in the annular domain $\widetilde{G}$ adjoining Γ along the whole of γ_i.

The representations (4) and (5) make it possible to establish additional properties of extremal functions under additional smoothness conditions on the data of the extremal problem. As an example we give the following theorem, which is a simple consequence of the representation (4).

Theorem 3. *In order that, under the conditions of Theorem 2, the modulus $|f^*(z)|$ be a function continuous up to γ, it is necessary and sufficient that $\rho(\zeta)$ coincide almost everywhere on γ with a continuous function.*

The question of the continuity of the function $f^*(z)$ itself up to γ is reduced by our representation (4) to the question of the continuity in a closed domain of the complex Green integral. Many works are devoted to this latter topic. Applying the results obtained in them, we can, after imposing additional smoothness conditions on $\rho(\zeta)$ and γ, obtain various results on the behavior of $f^*(z)$ in the closed domain adjoining γ. In particular, if γ is assumed to be an analytic arc, and $\rho(\zeta)$ and $\omega(\zeta)$ analytic functions of arc length on γ, then we obtain the possibility of analytic continuation of the extremal functions $f^*(z)$ and $\varphi^*(z)$ through the arc γ. The corresponding results were already given in ([3]), but there $\rho(\zeta) > 0$ was continuous on all of Γ, whereas here only the fulfillment of condition (1) is required of $\rho(\zeta)$ on Γ as a whole.

Theorem 4. *Suppose that, under the conditions of Theorem 1, $\omega(z)$ on each boundary contour $\gamma_i \subset \Gamma$ coincides almost everywhere with the boundary values of an analytic function $\omega_i(z)$ belonging to the class $E_{1/\rho,1}(\widetilde{G}_i)$ in a sufficiently narrow strip $\widetilde{G}_i \subset G$ adjoining γ_i, $i = 1, \ldots, n$. Then the extremal function $f^*(z)$ has the form*

$$f^*(z) = e^{i\alpha} \exp\left[\frac{1}{2\pi}\int_\Gamma \ln\rho(\zeta)\frac{\partial P(\zeta,z)}{\partial n}\,ds\right]\cdot\exp\sum_{k=1}^{N} -P(z,z_k). \qquad (6)$$

Thus, under the conditions of Theorem 4, one obtains a global representation of the extremal function $f^*(z)$. If, in addition, one assumes that $\omega(\zeta)$ is the boundary value of a function $\omega(z)$ meromorphic in G, then one can also write a global representation of the extremal difference $\omega(z)-\varphi^*(z)$, analogous to (6). In this case the total number of zeros of $f^*(z)[\omega(z)-\varphi^*(z)]$ does not exceed $N+n-2$, where N is the number of poles of $\omega(z)$, and n is the connectivity of the domain G.

As a consequence we indicate the following result: the extremal

the function $f^*(z)$ in the problem of $\sup\limits_{f\in D_\rho}|f(\alpha)|$, $\alpha\in G$, has the form (6), and the number of its zeros is $m\le n-1$.

In conclusion we present those dual pairs of classes of functions for which the study of extremal problems and of the properties of extremal functions was carried out by us by the same methods as for those considered above. A detailed listing of the results obtained is omitted for lack of space.

Class $E_{\rho,p}(G)$. $f(z)\in E_{\rho,p}(G)$, if

$$\lim_{j\to\infty}\int_{\Gamma^j}\left|\frac{f(z)}{\rho(z)}\right|^p ds\le\infty,$$

where Γ^j and $\rho(z)$ are the same as in the definition of the class $E_{\rho,1}(G)$.

Class $E^1_{\rho,p}(G)$. $f(z)\in E^1_{\rho,p}(G)$, if $f(z)\in E_{\rho,p}(G)$ and

$$\int_\Gamma\left|\frac{f(\zeta)}{\rho(\zeta)}\right|^p ds\le 1.$$

Class $D_{\rho,p}(G)$. $f(z)\in D_{\rho,p}(G)$, if $f(z)\in D(G)$ and

$$\int_{\dot G}\left|\frac{f(\zeta)}{\rho(\zeta)}\right|^p ds<\infty.$$

Class $D^1_{\rho,p}(G)$. $f(z)\in D^1_{\rho,p}(G)$, if $f(z)\in D(G)$ and

$$\int_{\dot G}\left|\frac{f(\zeta)}{\rho(\zeta)}\right|^p ds\le 1.$$

Class $A_{\rho,p}(G)$, $p>1$. $f(z)\in A_{\rho,p}(G)$, if $f(z)\in E_{\rho,1}(G)$ and

$$\int_{\dot G}\left|\frac{f(\zeta)}{\rho(\zeta)}\right|^p ds<\infty.$$

Class $A^1_{\rho,p}(G)$**.** $f(z) \in A^1_{\rho,p}(G)$, if $f(z) \in E_{\rho,1}(G)$ and

$$\int_\Gamma \left|\frac{f(\zeta)}{\rho(\zeta)}\right|^p ds \leq 1.$$

Class $A_{\rho,\infty}(G)$**.** $f(z) \in A_{\rho,\infty}(G)$, if $f(z) \in E_{\rho,1}(G)$ and if

$$\operatorname{vrai\,max}_{\zeta\in\Gamma} \left|\frac{f(\zeta)}{\rho(\zeta)}\right| < \infty.$$

Class $A^1_{\rho,\infty}(G)$**.** $f(z) \in A^1_{\rho,\infty}(G)$, if $f(z) \in E_{\rho,1}(G)$ and if

$$\operatorname{vrai\,max}_{\zeta\in\Gamma} \left|\frac{f(\zeta)}{\rho(\zeta)}\right| \leq 1.$$

The dual pairs of classes are formed by: $E^1_{\rho,p}$ and $E_{\rho,q}$; $A^1_{\rho,p}$ and $D_{\rho,q}$; $D'_{\rho,p}$ and $A_{\rho,q}$—in all these cases $\frac{1}{p} + \frac{1}{q} = 1$; further, $A^1_{\rho,\infty}$ and $D_{\rho,1}$; $D'_{\rho,1}$ and $A_{\rho,\infty}$.

The classes $A_{\rho,p}(G)$ and $D_{\rho,p}(G)$ are generalizations of the classes considered in work ([5]), where $\rho(\zeta) \equiv 1$ was taken. In addition, in work ([5]) these classes were introduced in a more formal way, not revealing the character of their admissible growth inside G.

Applying representation (6) and the global analogue of (5), and their analogues in other classes, to the particular case when G is the unit disk, one can obtain generalizations of the results on the representation of extremals given in works ([13,4,14,3]) (in these works one can find a detailed bibliography of preceding work). For multiply connected domains, a quite special case of such representations was indicated in work ([3]).

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Note: Figure translations are in progress. See original paper for figures.