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Abstract

Full Text

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On the Saturation Class for the Hölder Method of Summation of Fourier Series

(Presented by Academician S. L. Sobolev on 25 IV 1958)

Let $f(x)$ be a continuous 2π-periodic function and let

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

be its Fourier series. Denote by γ the summation process for this series defined by a sequence of constants γ_k^n $(k = 1, \ldots, n;\ n = 1, 2, \ldots)$, i.e., the γ-process of approximation of the function $f(x)$ by means of trigonometric polynomials of the sequence

$$P_n^{\gamma}(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \gamma_k^n (a_k \cos kx + b_k \sin kx).$$

Favard ([1]) posed the following problem: given a summation process γ, determine the class of functions for which this process is best.

Zamansky ([2,3]) refines Favard' s formulation of the problem and introduces the notion of a saturation class for a given summation method. He defines this notion in the following way. If there exists an increasing function $\varphi_\gamma(n)$ such that for every continuous 2π-periodic function $f(x)$, distinct from a constant, and every natural n we have

$$\max_x \varphi_\gamma(n)\, |P_n^{\gamma}(x) - f(x)| > a,$$

where a is a positive constant depending on f, and, moreover, there exist functions $f(x)$ for which

$$\max_x \varphi_\gamma(n)\, |P_n^{\gamma}(x) - f(x)| < b,$$

where $b > 0$ is another constant, also depending on f, then we shall say that the summation process γ is **saturated**. The **saturation class** corresponding to the process γ will be the set of continuous 2π-periodic functions, distinct from constants, such that

$$|P_n^\gamma(x) - f(x)| = O\left(\frac{1}{\varphi_\gamma(n)}\right).$$

Zamansky determined the saturation classes corresponding to certain summation processes (the Cesàro process, the Jackson process, the Vallée-Poussin process, etc.).

In his survey report at the 3rd All-Union Mathematical Congress, Favard indicated that it would be interesting to determine the saturation classes for other summation methods, in particular for the Hölder summation method H^r, where $r > 0$. This method is defined as follows ([4]). Let

$$\sum_{k=0}^{\infty} A_k \tag{1}$$

the given numerical series,

$$s_k = \sum_{\nu=0}^{k} A_\nu, \qquad (k = 0, 1, 2, ...) \tag{2}$$

are its partial sums. Then the Hölder sums of order $r > 0$ of the series (1) are called the sums

$$H_n^r = \sum_{k=0}^{n} C_n^k s_k \Delta^{n-k} \frac{1}{(k+1)^r}, \tag{3}$$

where s_k is defined by formula (2), and

$$\Delta^{n-k} \frac{1}{(k+1)^r} = \sum_{p=0}^{n-k} \frac{(-1)^p C_{n-k}^p}{(k+p+1)^r}.$$

It is not difficult to transform the sum (3) into the form

$$H_n^r = \sum_{k=0}^{n} h_{nk}^r A_k,$$

where

$$h^r_{n0} = 1, \qquad h^r_{nk} = nC^{k-1}_{n-1} \sum_{p=0}^{n-k} \frac{(-1)^p C^p_{n-k}}{(p+1)(p+k+1)^r} \qquad (k = 1, \dots, n). \quad (4)$$

Applying the Hölder summation method to the Fourier series of a continuous 2π-periodic function $f(x)$, we obtain a sequence of trigonometric polynomials

$$H^r_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} h^r_{nk} \left(a_k \cos kx + b_k \sin kx\right) \qquad (n = 0, 1, 2, \dots),$$

where the multipliers h^r_{nk} are defined by formula (4).

Lemma 1.

$$1 - h^r_{nk} = \frac{k \ln^{r-1} n}{n\Gamma(r)} (1 + \varepsilon^r_{nk}),$$

where $\lim_{n\to\infty} \varepsilon^r_{nk} = 0$ for fixed k and r.

Theorem 1. Let there be given a summation process γ, determined by a sequence of constants γ^n_k $(k = 1, \dots, n;\ n = 1, 2, \dots)$, and let there exist a non-negative function of the natural argument $\psi(n)$ such that $\lim_{n\to\infty} \psi(n) = 0$, and suppose that for every fixed k

$$1 - \gamma^n_k \sim c_k \psi(n),$$

where $c_k \geqq 0$ is a constant depending on k. Then the process γ is saturated with approximation of saturation order $\psi(n)$.

Corollary. The Hölder summation method of order $r > 0$ is saturated with approximation of saturation order

$$\frac{\ln^{r-1} n}{n}.$$

To find the saturation class belonging to the approximation process under consideration, i.e. to find the conditions satisfied by functions $f(x)$ such that

$$|H^r_n(x) - f(x)| = O\left(\frac{\ln^{r-1} n}{n}\right), \quad (5)$$

one must first find conditions under which

$$\mathcal{E}_n(f) = O\left(\frac{\ln^{r-1} n}{n}\right), \tag{6}$$

where $\mathcal{E}_n(f)$ is the best approximation of the function $f(x)$ by trigonometric polynomials of degree not exceeding n, since the functions satisfying (5) must be sought in the class of functions satisfying (6). For this latter class the following theorem of Zygmund type holds, following from Theorems 1 and 8 of the paper of S. B. Stechkin ([5]).

Theorem 2. In order that the best approximation $\mathcal{E}_n(f)$ of a continuous 2π-periodic function $f(x)$ by trigonometric polynomials of degree not exceeding n $(n = 2, 3, ...)$ satisfy the condition

$$\mathcal{E}_n(f) = O\left(\frac{\ln^{r-1} n}{n}\right),$$

it is necessary and sufficient that there exist a constant $A > 0$ such that, for all real values of x and for values of h from the interval $0 \le h \le \frac{1}{2}$, the condition

$$|f(x+h) + f(x-h) - 2f(x)| \le Ah|\ln h|^{r-1}.$$

Lemma 2. The following equalities hold:

$$H_n^r(x) - f(x) = \frac{1}{\pi}\int_{\alpha/n}^{1/4} \frac{\varphi(t)}{t^2}\sum_{k=1}^{n+1} b_{nk}^r \sin^2 kt\, dt + O\left(\frac{\ln^{r-1} n}{n}\right),$$

$$H_n^r(x) - f(x) =$$

$$= \operatorname{Re}\frac{1}{\pi\Gamma(r)}\int_{\alpha/n}^{1/4} \varphi(t)(1 - i\operatorname{ctg} t)\, dt \int_0^1 \left(\ln\frac{1}{n}\right)^{r-1} \left[u(e^{2it}-1)+1\right]^n du + O\left(\frac{\ln^{r-1} n}{n}\right),$$

$$H_n^r(x) - f(x) =$$

$$= \operatorname{Re}\frac{\ln^{r-1} n}{\pi n\Gamma(r)}\int_{\alpha/n}^{1/4} \varphi(t)(1 - i\operatorname{ctg} t)\, dt \int_0^n \left(1 - \frac{\ln v}{\ln n}\right)^{r-1} \left[1 + \frac{(e^{2it}-1)v}{n}\right]^n dv +$$

$$+ O\left(\frac{\ln^{r-1} n}{n}\right),$$

where α is any fixed positive number;

$$\varphi(t) = f(x+2t) + f(x-2t) - 2f(x); \tag{7}$$

$\operatorname{Re} F(t)$ denotes the real part of the complex function $F(t)$; the numbers b^r_{nk} are defined by the formula

$$b^r_{nk} = \frac{C^k_{n+1}}{n+1} \sum_{p=0}^{n+1-k} \frac{(-1)^p C^p_{n+1-k}}{(k+p)^{r-1}} \qquad (k = 1, \dots, n+1).$$

The saturation classes corresponding to the Hölder summation process of the second and third orders ($r = 2,\ r = 3$) are determined by the following theorems.

Theorem 3. In order that $\left|H^2_n(x) - f(x)\right| = O\left(\frac{\ln n}{n}\right)$, it is necessary and sufficient that the integral

$$\int_\varepsilon^{1/2} \left(1 - \frac{\ln t}{\ln \varepsilon}\right) \frac{f(x+t) + f(x-t) - 2f(x)}{t^2}\, dt$$

be bounded uniformly with respect to x and $\varepsilon > 0$.

Theorem 4. In order that $\left|H^3_n(x) - f(x)\right| = O\left(\frac{\ln^2 n}{n}\right)$, it is necessary and sufficient that the integral

$$\int_\varepsilon^{1/4} \frac{\varphi(t)}{t^2} \left[\left(1 - \frac{\ln t}{\ln \varepsilon}\right)^2 + \frac{2(C + \ln 2)\ln \frac{t}{\varepsilon} + C'}{\ln^2 \varepsilon}\right] dt$$

be uniformly bounded with respect to x and $\varepsilon > 0$. Here $\varphi(t)$ is defined by formula (7), and C and C' are absolute constants.

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References

[1] J. Favard, Bull. Sci. Math., **61**, 209 (1937).
[2] M. Zamansky, Ann. Sci. École Norm. Sup. (3), **66**, 19 (1949).
[3] M. Zamansky, Ann. Sci. École Norm. Sup. (3), **67**, 161 (1950).

[4] G. Hardy, *Divergent Series*, IL, 1951.
[5] S. B. Stechkin, Izv. Acad. Sci. USSR, Ser. Math., **15**, No. 3, 219 (1951).

Note: Figure translations are in progress. See original paper for figures.