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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**B. M. Bredikhin**

# FREE NUMERICAL SEMIGROUPS WITH POWER DENSITIES

*(Presented by Academician I. M. Vinogradov, 16 VIII 1957)*

1. Analysis of elementary proofs of the asymptotic law of distribution of prime numbers <sup>(1,2)</sup> shows that this law is a reflection only of the multiplicative and order properties of the semigroup of natural numbers and is not connected with the additive structure of the set of natural numbers. Indeed, passing to the consideration of semigroups of real numbers, we find that in such semigroups as well there is an asymptotic law of distribution of the generating elements of the semigroup, under a certain density of distribution of the elements of the semigroup on the number axis.

Let  $G$  be a multiplicative semigroup of real numbers  $\alpha \geq 1$  ( $1 \in G$ ), ordered according to their magnitude:

$$1 = \alpha_1 < \alpha_2 \leq \alpha_3 \leq \dots$$

The numbers  $\omega_1 \leq \omega_2 \leq \omega_3 \leq \dots$  ( $\omega_i \in G$ ;  $\omega_i > 1$ ) are generating elements of the semigroup  $G$ . The existence of generating elements in the semigroup  $G$  means that each element  $\alpha \in G$  is uniquely written in the form  $\alpha = \omega_1^{x_1} \omega_2^{x_2} \dots$ , where the  $x_i$  are nonnegative integers, with only finitely many  $x_i \neq 0$ .

We shall call the semigroup  $G$  a **free numerical semigroup**. In what follows we consider only such free semigroups which have no limit points at a finite distance (in the semigroup  $G$  all elements are counted, distinct in their representation, regardless of their possible equality in magnitude).

Put

$$\nu_G(x) = \sum_{\alpha \leq x, \alpha \in G} 1, \quad \pi_G(x) = \sum_{\omega \leq x, \omega \in G} 1.$$

If there exists

$$\lim_{x \rightarrow \infty} (\nu(x)/x^\theta) = C,$$

where  $\theta > 0$ ,  $C > 0$ , then we shall call  $C$  the **power  $\theta$ -density** of the semigroup  $G$ . In the case  $\theta = 1$ , the concept of power density coincides with the concept of natural density.

In this note we briefly present an elementary solution of the problem of finding the asymptotics of  $\pi_G(x)$  from the given asymptotics of  $\nu_G(x)$ . The proofs are based on methods developed in the works of Euyb <sup>(3)</sup> and Breusch <sup>(4)</sup>, and relying on the ideas of Selberg and Shapiro.

**2.** Define on the semigroup  $G$  the functions:

$$\mu_G(\alpha) = \begin{cases} 1, & \text{if } \alpha = 1; \\ (-1)^k, & \text{if } \alpha = \omega_1 \cdots \omega_k; \\ 0, & \text{if } \omega_i^2/\alpha; \end{cases}$$

$$\Lambda_G(\alpha) = \begin{cases} \log \omega, & \text{if } \alpha = \omega^x \ (x > 0), \\ 0, & \text{if } \alpha \neq \omega^x; \end{cases} \quad \psi_G(x) = \sum_{\alpha \leq x} \Lambda_G(\alpha).$$

**Lemma** (generalized Selberg lemma). *Let  $G$  be a free numerical semigroup having power  $\theta$ -density, and suppose*

$$\nu_G(x) = Cx^\theta + O(x^{\theta_1}), \quad \text{where } \theta_1 < \theta. \quad (1)$$

Then

$$\psi_G(x) \log x + \sum_{\alpha \leq x} \Lambda_G(\alpha) \psi_G\left(\frac{x}{\alpha}\right) = \frac{2}{\theta} x^\theta \log x + O(x^\theta). \quad (2)$$

**Proof.** Consider the Möbius transformation: if

$$f(x) = \sum_{\alpha \leq x} h\left(\frac{x}{\alpha}\right) \log x,$$

then

$$\sum_{\alpha \leq x} \mu_G(\alpha) f\left(\frac{x}{\alpha}\right) = h(x) \log x + \sum_{\alpha \leq x} \Lambda_G(\alpha) h\left(\frac{x}{\alpha}\right). \quad (3)$$

Putting

$$h(x) = \theta \psi_G(x) - x^\theta + 1 + \frac{C_0}{C},$$

where  $C_0$  is a constant determined by the subsemigroup  $G$ , and using inequality (1), we derive the estimate

$$\begin{aligned} & h(x) \log x + \sum_{\alpha \leq x} \Lambda_G(\alpha) h\left(\frac{x}{\alpha}\right) \\ &= \theta \psi_G(x) \log x + \theta \sum_{\alpha \leq x} \Lambda_G(\alpha) \psi_G\left(\frac{x}{\alpha}\right) - 2x^\theta \log x + O(\psi_G(x)). \end{aligned} \quad (4)$$

On the other hand,

$$\sum_{\alpha \leq x} \mu_G(\alpha) f\left(\frac{x}{\alpha}\right) = O(x^\theta). \quad (5)$$

From (3), (4), and (5) the generalized Selberg inequality follows.

3. Introduce the function

$$\vartheta_G(x) = \sum_{\omega \leq x} \log \omega.$$

**Theorem 1** (asymptotic law). *Let the conditions of the generalized Selberg lemma be satisfied for the free numerical subsemigroup. Then*

$$\pi_G(x) \sim \frac{1}{\theta} \frac{x^\theta}{\log x}. \quad (6)$$

**Proof.** It is enough to prove that

$$\vartheta_G(x) \sim \frac{1}{\theta} x^\theta, \quad (7)$$

since the asymptotic equalities (6) and (7) are equivalent.

Let  $R$  be the interval  $\left(\log x, \frac{x}{\log x}\right)$ . Estimate  $\sum_{\alpha \in R} \vartheta_G\left(\frac{x}{\alpha}\right)$ . With the help of elementary properties of the function  $\vartheta_G(x)$ , one obtains the lower estimate:

$$\sum_{\alpha \in R} \vartheta_G\left(\frac{x}{\alpha}\right) > (1-h)Cx^\theta \log x, \quad (8)$$

where  $h < 1$  is a sufficiently small constant.

From the generalized Selberg inequality, taking into account that

$$\overline{\lim}_{x \rightarrow \infty} \frac{\vartheta_G(x)}{x^\theta} = \frac{1}{\theta} + \gamma, \quad \underline{\lim}_{x \rightarrow \infty} \frac{\vartheta_G(x)}{x^\theta} = \frac{1}{\theta} - \gamma,$$

and, assuming that  $\gamma \neq 0$ , we derive the upper estimate:

$$\sum_{\alpha \in R} \vartheta_G\left(\frac{x}{\alpha}\right) < (1-h)Cx^\theta \log x. \quad (9)$$

But the estimates (8) and (9) are contradictory. Consequently,  $\gamma = 0$ , which proves formula (7).

4. Theorem 1, proved under very general conditions, contains a number of special theorems that are obtained by considering special semigroups. We give several examples of specializations of Theorem 1.

- 1) Theorem 1 is stated literally for numerical semigroups with bases (in such semigroups the logarithms of the basic-generating-elements are linearly independent).

In particular, by considering the semigroup of all natural numbers, we obtain the asymptotic law of distribution of prime numbers.

- 2) Let  $\pi(x)$  be the number of all Gaussian primes lying in the circle of radius  $x$ . Theorem 1 and the estimate for the number of lattice points inside a circle (5) yield the known result:

**Theorem 2.**

$$\pi(x) \sim 2 \frac{x^2}{\log x}.$$

Analogous results can be obtained in all those cases where semigroups of Gaussian numbers with power densities are formed from the prime Gaussian numbers of the first quadrant.

- 3) Theorem 1 is applied to the solution of one class of inverse problems in additive number theory (another class of problems was considered in the works of G. A. Freiman <sup>(6)</sup>).

Let

$$a_1, a_2, \dots, a_n, \dots \tag{10}$$

be a monotone nondecreasing sequence of positive numbers; let  $n(u)$  be the number of terms of the sequence (10) not exceeding  $u$ ; and let  $q(u)$  be the number of solutions of the inequality

$$n_1 a_1 + n_2 a_2 + \dots \leq u$$

in nonnegative integers.

**Theorem 3.** If

$$q(u) = C e^{\theta u} + O(e^{\theta_1 u}),$$

where  $\theta > 0$ ,  $\theta_1 < 0$ , then

$$n(u) \sim \frac{1}{\theta} \frac{e^{\theta u}}{u}.$$

- 4) From Theorem 1 there follows the asymptotic law of distribution of prime ideals in semigroups of ideals of algebraic fields, in particular, Shapiro's known result <sup>(7)</sup>.
5. The study of the arithmetic of free numerical semigroups by elementary methods may prove useful in connection with those problems that arise in the general theory of characters of numerical semigroups <sup>(8, 9)</sup> and that are also solved most naturally when we consider semigroups of real numbers <sup>(10)</sup>.

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## References

1. A. Selberg, Ann. Math., (2) **50** (1949).
2. A. G. Postnikov, N. P. Romanov, Uspekhi Mat. Nauk, **10**, 4 (1955).
3. R. Ayoub, Canad. J. Math., **7**, No. 1 (1955).
4. R. Breush, Duke Math. J., **21**, No. 1 (1954).
5. I. M. Vinogradov, Izv. AN SSSR, ser. fiz.-matem., No. 3, 313 (1932).
6. G. A. Freiman, Izv. AN SSSR, ser. matem., **19**, No. 4 (1955).
7. H. Shapiro, Comm. Pure and Appl. Math., **2** (1949).
8. N. G. Chudakov, Tr. 3-go Vsesoyuzn. matem. s" ezda, 1, Izd. AN SSSR, 1956.
9. B. M. Bredikhin, Tr. 3-go Vsesoyuzn. matem. s" ezda, 1, Izd. AN SSSR, 1956.
10. N. G. Chudakov, B. M. Bredikhin, Ukr. matem. zhurn., **8**, No. 4 (1956).

*Note: Figure translations are in progress. See original paper for figures.*

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