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Abstract

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AN EXISTENCE AND UNIQUENESS THEOREM FOR THE SOLUTION OF A BOUNDARY-VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician V. I. Smirnov, 10 VII 1958)

Let the following be given:

1) A system of differential equations

$$y' = f(t, y), \tag{1}$$

where $y(t)$ is the unknown n-dimensional vector, and $f(t, y)$ is a vector-function.

With respect to the functions $f_i,\ i = 1, 2, \dots, n$, it is assumed that they take real values, are continuous, and have continuous partial derivatives

$$f_{ik}(t, y) = \frac{\partial f_i(t, y_1, \dots, y_n)}{\partial y_k}, \qquad i, k = 1, \dots, n,$$

in a domain G of the real $(n + 1)$-dimensional Euclidean space of the variables $t, y_1, y_2, \dots, y_n$.

2) Boundary conditions

$$\sum_{m=0}^{\mu} \alpha_m y(t_m) = b, \qquad t_0 \le t_1 \le \dots \le t_\mu, \tag{2}$$

where $\alpha_m,\ m = 0, 1, \dots, \mu$, are given constant matrices of order n; b is a given constant vector.

3) The vector $Y(t)$ has a continuous derivative $Y'(t)$ for $t \in [t_0, t_\mu]$, and for $t \in [t_0, t_\mu]$ the point $(t, Y(t)) \in G^*$, where $G^* \subset G$ and is convex in all $y_i,\ i = 1, 2, \dots, n$,

$$\sum_{m=0}^{\mu} \alpha_{mY}(t_m) = B; \tag{3}$$

hence it also follows that the interval $[t_0, t_\mu]$ is contained in the projection of the domain G onto the t-axis.

4) The matrix $U(t)$ belongs to the class C' for $t \in [t_0, t_\mu]$ and is nonsingular for every $t \in [t_0, t_\mu]$ (cf. (1)).

Put:

1) $J(t, y)$ is the Jacobi matrix defined by the formulas

$$\{J(t, y)\}_{ik} = f_{ik}(t, y), \qquad i, k = 1, 2, \dots, n.$$

2) $Q(t, y)$ is the matrix defined by the formula

$$Q(t, y) = U^{-1}(t) \cdot J(t, y) \cdot U(t) - U^{-1}(t) \cdot \frac{d}{dt} U(t). \tag{4}$$

3) $P(t)$ is a square matrix of order n, continuous for $t \in [t_0, t_\mu]$, and $M_P(t, t_0)$ is its matricant ([3])

$$M_P(t, t_0) = E + \int_{t_0}^{t} P(u)\, du + \int_{t_0}^{t} P(u) \int_{t_0}^{u} P(u_1)\, du_1\, du + \cdots,$$

where E is the identity matrix of order n.

4) D_P is the matrix defined by the formula

$$D_P = \sum_{m=0}^{\mu} \alpha_m \cdot U(t_m) \cdot M_P(t_m, t_0).$$

5) If $\det D_P \neq 0$, then the matrix

$$G_P(t, \xi) = \frac{1}{2} M_P(t, t_0) \cdot D_P^{-1} \left\{ \sum_{m=0}^{\mu} [\operatorname{sign}(t - \xi) - \operatorname{sign}(t_m - \xi)]\, \alpha_m \cdot U(t_m) \cdot M_P(t_m, t_0) \right\} \cdot M_P^{-1}(\xi, t_0),$$

$\xi \neq t$ and $\xi \neq t_m$, $m = 0, 1, \dots, \mu$, will be called the Green matrix corresponding to the boundary-value problem

$$z' = P(t) \cdot z + \varphi(t); \tag{5}$$

$$\sum_{m=0}^{\mu} \alpha_m \cdot U(t_m) \cdot z(t_m) = 0; \tag{6}$$

the solution of problem (5)–(6) has the form

$$z(t) = \int_{t_0}^{t_\mu} G_P(t, \xi) \cdot \varphi(\xi) \, d\xi.$$

6) $\tau(t)$ is the residual vector defined by the formula

$$\tau(t) = Y'(t) - f[t, Y(t)].$$

Then the following theorem holds:

Theorem. *Let:*

a) $\det D_p \neq 0$;

b) in the domain G^*

$$\|G_P(t, \xi)\{Q(\xi, y) - P(\xi)\}\| \leq K(t, \xi), \tag{7}$$

where $K(t, \xi)$ is a real, bounded, nonnegative function, continuous or having discontinuities in the square $t_0 \leq t, \xi \leq t_\mu$ on the same straight lines as $G_P(t, \xi)$; here and below the norm is understood in the sense of the first norm of a matrix and a vector ([2]);

c)

$$\int_{t_0}^{t_\mu} \int_{t_0}^{t_\mu} K^2(t, \xi) \, d\xi \, dt < 1; \tag{8}$$

d) $u(t)$ is the solution of the integral equation

$$u(t) = \int_{t_0}^{t_\mu} K(t, \xi) \cdot u(\xi) \, d\xi + \|\varepsilon_0(t)\|, \tag{9}$$

where $\varepsilon_0(t)$ is the solution of the boundary-value problem

$$\varepsilon_0' = P(t) \cdot \varepsilon_0 - U^{-1}(t) \cdot \tau(t),$$

$$\sum_{m=0}^{\mu} \alpha_m \cdot U(t_m) \cdot \varepsilon_0(t_m) = b - B \tag{10}$$

or, whence

$$\varepsilon_0(t) = -\int_{t_0}^{t_\mu} G_P(t, \xi) \cdot U^{-1}(\xi) \cdot \tau(\xi) \, d\xi + M_P(t, t_0) \cdot D_P^{-1} \cdot (b - B); \tag{11}$$

d) the domain determined by the inequalities

$$t_0 \le t \le t_\mu, \qquad \|Y(t) - y\| \le \|U(t)\| \cdot u(t),$$

lies in G^*.

Then on the interval $[t_0, t_\mu]$ there exists a unique solution $y(t)$ of system (1) with boundary conditions (2) such that

$$\|Y(t) - y(t)\| \le \|U(t)\| \cdot u(t), \qquad t_0 \le t \le t_\mu. \tag{12}$$

Remark 1. The conditions of the theorem include four parameters: the matrices $U(t)$, $P(t)$, the vector $Y(t)$, and the domain G^*. It is advantageous to choose the matrix $U(t)$ from the condition that the matrix $Q(t, y)$ change its value as little as possible in the section of the domain G^* by the plane $t = \text{const}$, $t_0 \le t \le t_\mu$. It is advantageous to determine the matrix $P(t)$ from the condition that the left-hand side of (7) be as small as possible.

If in the domain G^*

$$\{a(t)\}_{ik} \le \{Q(t, y)\}_{ik} \le \{A(t)\}_{ik},$$

then one usually sets

$$P(t) = \frac{1}{2}[A(t) + a(t)].$$

The conditions of the theorem are such that, if problem (1)–(2) has a solution in G, then there can always be found such $U(t)$, $P(t)$, $Y(t)$, and G^* that all the conditions will be satisfied.

Remark 2. Let the interval $[t_0, T]$, where $t_\mu \le T$, be contained in the projection of the domain G^* onto the t-axis.

Define the matrix $G^*_P(t, \xi)$ as follows:

$$G^*_P(t, \xi) = \begin{cases} G_P(t, \xi), & t_0 \le t \le t_\mu, \quad t_0 < \xi < t_\mu; \\ 0, & t_0 \le t \le t_\mu, \quad t_\mu < \xi < T; \\ M_P(t, t_\mu) \cdot G_P(t_\mu, \xi), & t_0 < \xi < t_\mu, \quad t_\mu \le t \le T; \\ M_P(t, \xi), & t_\mu < \xi < t \le T; \\ 0, & t_\mu \le t < \xi < T. \end{cases}$$

Define the function $K^*(t, \xi)$ by the formula

$$\|G^*_P(t, \xi) \cdot \{Q(\xi, y) - P(\xi)\}\| \le K^*(t, \xi).$$

Then, under the condition

$$\int_{t_0}^{T}\int_{t_0}^{T} K^{*2}(t,\xi)\,d\xi\,dt < 1,$$

the theorem guarantees the existence of a solution of problem (1)–(2) for $t \in [t_0, T]$.

Remark 3. Inequality (12) may be taken as a formula for estimating the error of an approximate solution of problem (1)–(2) on the interval $[t_0, t_\mu]$, if the vector $Y(t)$ is taken as the approximate solution.

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CITED LITERATURE

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3. F. R. Gantmakher, *Theory of Matrices*, Moscow, 1954.

Note: Figure translations are in progress. See original paper for figures.