MATHEMATICS

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Abstract

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ON THE UNSOLVABILITY OF CERTAIN PROBLEMS OF TOPOLOGY

1. This note is devoted to strengthening and constructively refining the results formulated in the note ([3]). There the question concerned the problem of homeomorphism of polyhedra specified by schemes of their triangulations. Here we shall be concerned mainly with these schemes themselves, i.e., with the so-called "complexes."

The foundations of the purely combinatorial theory of complexes were laid in the works of Newman ([5]) and Alexander ([4]). In these works, in particular, the concept of "combinatorial equivalence" of complexes is defined.

Since complexes are constructive objects and may be regarded as words in a suitable alphabet, there arises the general problem of the combinatorial equivalence of complexes, consisting in finding an algorithm,* which determines for any pair of complexes whether they are combinatorially equivalent. This problem is a special case of the general problem of recognizing a given binary relation between complexes, formulated as follows. Let a binary relation $\mathfrak{R}$ between complexes be defined. It is required to construct an algorithm which determines, for any pair of complexes, whether they are related by the relation $\mathfrak{R}$. This problem turns out to be unsolvable for a certain rather broad class of binary relations between complexes, including combinatorial equivalence.

Along with the general problem of recognizing the relation $\mathfrak{R}$, there arise various special problems of recognizing this relation, formulated for complexes subject to one or another restriction. For example, by fixing a natural number n, one may pose the problem of recognizing the relation $\mathfrak{R}$ among complexes of dimension not exceeding n. One may also pose the problem of recognizing the relation $\mathfrak{R}$ among n-dimensional manifolds, where the term "manifold" is understood in the sense of Alexander ([4]).

Another natural restriction imposed on the complexes under consideration is the fixing of one of them. In this case there arises the problem of recognizing the relation $\mathfrak{R}$ to a given complex K, consisting in finding an algorithm which determines, for any complex, whether it has the relation $\mathfrak{R}$ to the complex K.

Such restrictions may also be combined with one another.

2. An essential role in the formulation of our main result is played by the concept of the fundamental group of a connected complex. To every connected complex K there may be assigned in a completely definite way a group calculus, called the fundamental group of the complex K (see (2), VII, §46).

Let us agree to say that a complex K is *related* to a complex L in the following two cases: when K and L are connected and their fundamental groups are isomorphic; when at least one of these complexes is not connected.

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* Here and below the term "algorithm" is understood in its exact sense, as a normal algorithm.

Let us also agree on the following terminology. Let $\mathfrak{R}$ and $\mathfrak{S}$ be binary relations between complexes. We shall say that $\mathfrak{R}$ is **stronger** than $\mathfrak{S}$ if a complex K has relation $\mathfrak{S}$ to a complex L whenever K has relation $\mathfrak{R}$ to L. We shall say of a binary relation $\mathfrak{T}$ between complexes that it is **contained between** $\mathfrak{R}$ and $\mathfrak{S}$ if $\mathfrak{R}$ is stronger than $\mathfrak{T}$, and $\mathfrak{T}$ is stronger than $\mathfrak{S}$.

We can now formulate our main results as follows.

Theorem. *For every natural number n greater than three, one can specify a connected closed n-dimensional manifold M^n such that, for any binary relation between complexes $\mathfrak{R}$ contained between combinatorial equivalence and kinship, the problem of recognizing the relation $\mathfrak{R}$ to M^n among n-dimensional manifolds is undecidable.*

Corollary 1. *For every $n > 3$ and every $\mathfrak{R}$ contained between combinatorial equivalence and kinship, the problem of recognizing $\mathfrak{R}$ among n-dimensional manifolds is undecidable.*

Corollary 2. *For every $n > 3$ and every $\mathfrak{R}$ contained between combinatorial equivalence and kinship, the problem of recognizing $\mathfrak{R}$ among complexes of dimension not exceeding n is undecidable.*

Corollary 3. *For every $\mathfrak{R}$ contained between combinatorial equivalence and kinship, the general problem of recognizing $\mathfrak{R}$ is undecidable.*

3. The proof of the theorem formulated above can be carried out analogously to the proof of theorem 1 of note (3). The difference will consist in the fact that now $\mathfrak{M}(P_1 * ... * P_m * r)$ will be not a polyhedron, but a complex (of a certain triangulation of the former $\mathfrak{M}(P_1 * ... * P_m * r)$). This complex, which is a connected closed 4-dimensional manifold in the sense of Alexander, is constructed directly from the given words P_i and the number r. Conditions U1-U6 and lemma 1 remain valid, with the words "homeomorphic," "homeomorphic to" replaced by "combinatorially equivalent," "combinatorially equivalent to." The manifold M^4 is defined by the equality

$$M^4 = \mathfrak{M}(*^k 0), \qquad (1)$$

where k is chosen as in note ([3]). For $n > 4$ the manifold M^n is defined by the equality

$$M^n = M^4 \times S^{n-4},$$

where S^h denotes the h-dimensional sphere in the sense of Alexander, and $\times$ is the sign of topological multiplication of complexes (see ([2]), p. 68). The number r is chosen as in note ([3]).

Let now $\mathfrak{R}$ be a binary relation between complexes, contained between combinatorial equivalence and kinship.

If the group with generating elements $\alpha_1, \dots, \alpha_r$, defined by the system of relations

$$R_i rightarrow \Lambda \qquad (i = 1, \dots, k)$$

in the alphabet Γ_r, is the trivial group, then, according to (1) and lemma 1, the manifold $\mathfrak{M}(R_1 * \dots * R_k *^{r+1} r)$ is combinatorially equivalent to the manifold M^4, and therefore has relation $\mathfrak{R}$ to M^4.

If, however, this group is not trivial, then, according to U1, the connected manifold $\mathfrak{M}(R_1 * \dots * R_k *^{r+1} r)$ is not akin to the connected manifold M^4, and therefore does not have relation $\mathfrak{R}$ to M^4.

It follows from this that, with the aid of any algorithm recognizing for an arbitrary 4-dimensional manifold whether it has relation $\mathfrak{R}$ to M^4, one could construct an algorithm recognizing, for every (r, k)-group, whether it is

whether it is the identity. Meanwhile the latter algorithm is impossible by virtue of the way r and k were chosen. Thus the theorem has been proved for $n = 4$. The passage to any n greater than four presents no difficulty.

4. In note ([3]) the discussion concerned homeomorphism of polyhedra. The classical concept of "homeomorphism," however, is not exact, in view of the vagueness of the expression "there exists a mapping," which figures in the generally accepted definition of this concept. We believe that the proper clarification of this concept within the framework of constructive mathematics, to which we adhere, should consist in the following.

The concept of a "projection spectrum" introduced by P. S. Aleksandrov ([1]) admits a natural constructive refinement, which leads to the concept of a "constructive spectrum." Constructive spectra are constructive objects written by words in a certain alphabet. The concepts of a "constructive continuous" and a "constructive topological" mapping of a constructive spectrum onto a constructive spectrum are naturally defined. We say of two constructive spectra that they are constructively homeomorphic if a constructive topological mapping of one of them onto the other can be constructed.

"Polyhedra" are defined as constructive spectra of a certain special kind. Every complex determines a polyhedron, called the "body" of this complex. We say of two complexes that they are constructively homeomorphic if their bodies are constructively homeomorphic.

The relation of constructive homeomorphism of complexes turns out to be contained between combinatorial equivalence and kinship. Using this and taking into account that the body of every connected closed manifold in the sense of Aleksandrov is a manifold in the sense of Poincaré and Veblen, we obtain theorem 1 of note ([3]) as a consequence of the theorem formulated in item 2.

5. The concept of homotopy equivalence also admits a natural constructive refinement. In this way there arises the concept of "constructive homotopy equivalence" of constructive projection spectra, which is then transferred to complexes. The relation of constructive homotopy equivalence of complexes turns out to be contained between combinatorial equivalence and kinship, which gives theorem 2 of note ([3]).
6. The relation of combinatorial equivalence itself is contained between combinatorial equivalence and kinship, since it is stronger than kinship. Therefore, in our theorem and in its corollaries one may take combinatorial equivalence as $\mathfrak{R}$. Thus, the general problem of combinatorial equivalence is unsolvable.

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Note: Figure translations are in progress. See original paper for figures.