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Abstract

Full Text

Mathematics

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ON CONFORMAL MAPPINGS OF RINGS AND ON THE BASIC BASIS OF THE SPACE OF FUNCTIONS ANALYTIC IN AN ELEMENTARY NEIGHBORHOOD OF AN ARBITRARY CONTINUUM

(Presented by Academician A. N. Kolmogorov, 27 I 1958)

1. Let K be a bounded continuum in the z-plane, not separating the plane, and let l_ρ be its level lines, i.e. the images of the circles $|w| = \rho$ under the schlicht mapping $w = \varphi(z)$ of the exterior of K onto the disk $|w| > 1$, $\varphi(\infty) = \infty$. Then, as was first clarified by S. N. Bernstein ([1]), the interiors of the curves l_ρ are natural neighborhoods of the continuum from the point of view of uniform approximation, on it, of analytic functions by means of polynomials. In other words, from a certain rate of best polynomial approximations on K one can conclude that the function is analytic inside l_ρ, and conversely. This fact is closely connected with another: in the space of functions analytic inside l_ρ there is a basis consisting of polynomials, the expansions with respect to which converge on the continuum with the rate of best approximations; such a basis is formed, as is known, by the Faber polynomials ([2,3]).

Now let a simply connected domain G contain K, but in general not be the interior of l_ρ. Consider the class A_G^K of all functions $f(z)$ analytic in G, and introduce the norm

$$\|f\| = \max_{z\in K} |f(z)|.$$

From the theorem of S. N. Bernstein cited above it follows that it is in general unreasonable to approximate an arbitrary function of the class A_G^K by polynomials. Generalizing G. Faber' s theorem, we shall show, however, that it is always possible to define a sequence $e_0(z), \ldots, e_n(z), \ldots$ of functions analytic in G and linearly independent for polynomials

$$\sum_{k=0}^{n} a_k e_k(z),$$

for which a proposition is valid (Theorem 3) analogous to S. N. Bernstein' s theorem. Moreover, the functions $e_n(z)$ form a basis of the space A_G^K, and this basis is "most efficient" : whatever basis is taken in the space A_G^K, the expansions with respect to it of any function of the space cannot converge on the continuum faster (in the sense of a progression with a smaller denominator) than the expansions with respect to $e_n(z)$. Finally, from the polynomials

$$\sum_{k=0}^{n} a_k e_k(z)$$

with suitable n, $n \simeq \log \frac{1}{\varepsilon}$, one can choose an "asymptotically" minimal ε-net in the compact class of functions $f(z) \in A_G^K$ such that $|f(z)| \leq M,\ z \in G$, for any $M > 0$.

The author considers it necessary to emphasize that the present note arose as a result of his work on the problem of the asymptotics of the ε-entropy of analytic functions posed by A. N. Kolmogorov.

2. On doubly connected domains

Let G be a certain doubly connected domain with boundary continua K and Γ. We orient the domain G, calling one of the boundary continua, for example K, "inner," the other "outer," and in accordance with this write $G = (K, \Gamma)$. By $\Gamma(K)$ we denote the simply connected domain with boundary Γ and containing K; the notation $K(\Gamma)$ has an analogous meaning. It is known that every doubly connected domain $G = (K, \Gamma)$ with nondegenerate inner continuum K can be conformally and univalently mapped onto a circular annulus $1 < |w| < R$ with preservation of orientation, where the mapping function $w = \varphi(z)$ (up to a factor $e^{i\theta}$) and the number R $(1 < R \leq +\infty)$, called the modulus of the domain G, are determined by the domain itself uniquely.

Theorem 1. *For any doubly connected domain $G = (K, \Gamma)$ the function $w = \varphi(z)$ can be represented in the form $\varphi(z) = \varphi^2(\varphi^1(z))$, where the functions $t = \varphi^1(z)$ and $w = \varphi^2(t)$ are univalent, respectively, in the simply connected domains $\Gamma(K)$ and $K^1(\Gamma^1)$. Here the function $t = \varphi^1(z)$ is determined by the domain G uniquely up to an arbitrary fractional-linear transformation of the plane (t).*

It is easy to see that Theorem 1 can be expressed otherwise:

Theorem 1′. *Let B be an arbitrary simply connected domain and K an arbitrary continuum, $K \subset B$. The domain B can be conformally transformed into the interior of some level line of the transformation by the same transformation of the continuum. The indicated transformation is unique up to a repeated linear transformation.*

For the proof of Theorem 1′ let us first map the domain B onto the disk $|z_0| < r_0$, $z_0 = f_0(z)$. Let $K_0 = f_0(K)$. We may suppose that K_0 does not separate the plane. Let $\widetilde{K}_0$ be the inverse of K_0 with respect to the circle $|z_0| = r_0$. Map conformally the domain complementary to $\widetilde{K}_0$ onto the disk $|z_1| < r_1$, $z_1 = f_1(z_0)$, $f_1(0) = 0$, $f_1'(0) = 1$. Let $K_1 = f_1(K_0)$. Let $\widetilde{K}_1$ be the inverse of

K_1 with respect to $|z_1| = r_1$. Next the domain complementary to $\widetilde{K}_1$ is mapped onto the disk $|z_2| < r_2$, $z_2 = f_2(z_1)$, $f_2(0) = 0$, $f_2'(0) = 1$. Here $K_2 = f_2(K_1)$, and so on. Put

$$\varphi_n^1(z) = f_n(f_{n-1}(\dots(f_0(z)))).$$

Applying the general theorems on univalent functions, we prove that $\varphi_n^1(z) \to \varphi^1(z)$. The proof that $\varphi^1(B)$ is the interior t_R for $\varphi^1(K)$ is obtained by means of the symmetry principle and the general theorems on convergence.

3.

In what follows, K denotes an arbitrary continuum in the plane (z), degenerate neither into a point nor into the whole plane. By D_q $(q = 0, 1, \dots)$ we denote the sequence of domains adjacent to K.

We shall call an "elementary neighborhood of the continuum K" an arbitrary domain $G \supset K$ of such a kind that inside each of the domains D_q there is located no more than one component of the boundary Γ of the domain G. If G is an elementary neighborhood of K, then $\Gamma_q = \Gamma \cap D_q$. Those q for which Γ_q is empty are henceforth excluded from consideration. Put $G_q = G \cap D_q$ and $K_q = K \cap [G_q]$. Obviously, each of the domains $G_q = (K_q, \Gamma_q)$ is doubly connected. Let the modulus of G_q be equal to R_q.

Theorem 2. *Whatever the continuum K and whatever its elementary neighborhood G, there exists a double sequence of functions* $e_{00}(z) = 1$, $e_{qn}(z)$ $(n = 1, 2, \dots)$ *such that:*

I. *All functions* $e_{qn}(z)$ *are analytic and single-valued in the domain G; more precisely, for any q all* $e_{qn}(z)$ *are analytic in the simply connected domain* $\Gamma_q(K_q)$.

II. *Every function* $f(z)$*, regular in G, has an expansion*

$$f(z) = a_{00} + \sum_q \sum_{n=1}^{\infty} a_{qn} e_{qn}(z),$$

converging uniformly and absolutely "inside" G.

III. *The expansion II is unique:* $a_{qn} = a_{qn}(f)$.

IV. The functions $e_{qn}(z)$ and the coefficients a_{qn} possess the following properties (in special cases they can be strengthened):

1°. $\|e_{qn}\| < C(\delta)(1+\delta)^n$.

2°. $\sup\limits_{z \in G} |e_{qn}(z)| < C(\delta)(1+\delta)^n R_q^n$.

3°. $|a_{qn}| < M_q C(\delta)(1+\delta)^n R_q^{-n}, \quad M_q = \sup\limits_{z \in G_q} |f(z)|$.

4°. $|a_{qn}| < C_q \|f\|$, if the Hausdorff length of K_q is $< +\infty$.

Here $\delta > 0$ is arbitrarily small and the constants $C(\delta)$ and C_q depend only on K and G; q and n are arbitrary; the case $R_q = +\infty$ is not formally excluded.

Lemma. Let G be a simply connected domain with boundary K, containing the point $z = 0$ and not containing ∞. Let the function $w = \varphi(z)$ map G one-to-one onto the circle $|w| > 1$ in such a way that $\varphi(0) = \infty$. By l_ρ denote the corresponding level lines. Consider the functions

$$\varphi_n(z) = \frac{1}{2\pi i}\int_{l_\rho} \frac{|\varphi(\zeta)|}{\zeta - z}\, d\zeta = \frac{\beta_n^{(n)}}{z^n} + \frac{\beta_{n-1}^{(n)}}{z^{n-1}} + \cdots + \frac{\beta_1^{(n)}}{z}, \qquad z \in l_\rho(K).$$

Then every function $f(z)$, analytic inside some l_ρ and such that $f(\infty) = 0$, has a unique expansion

$$f(z) = \sum_{n=1}^{\infty} a_n \varphi_n(z),$$

where the coefficients are determined by the formulas

$$a_n = \frac{1}{2\pi i}\int_{l_{\rho'}} f(\zeta)\,\frac{\varphi'(\zeta)}{[\varphi(\zeta)]^{n+1}}\, d\zeta, \qquad \rho' < \rho.$$

In the annulus $1 < |w| < \rho$ the function $f(\varphi^{-1}(w))$ has a Laurent expansion. Consequently, in the domain (K, l_ρ),

$$f(\zeta) = \sum_{n=-\infty}^{+\infty} a_n [\varphi(\zeta)]^n, \qquad a_n = \frac{1}{2\pi i}\int_{|w|=\rho'} f(\varphi^{-1}(w)) w^{-n-1}\, dw.$$

Multiply by $\dfrac{1}{2\pi i}\dfrac{1}{\zeta - z}$, where $z \in l_{\rho'}(K)$, and integrate with respect to $\zeta \in l_{\rho'}$. Since $f(\zeta)/(\zeta - z)$, as a function of ζ, has at ∞ a zero of at least second order, its residue with respect to ∞ is 0. Therefore we obtain

$$f(z) = \sum a_n \varphi_n(z).$$

It remains to note that $\varphi_n(z) \equiv 0$ for $n \leq 0$. Indeed, denoting $\varphi^{-1} = \psi$, we have

$$\varphi_n(z) = \frac{1}{2\pi i}\int_{|w|=\rho} \frac{\psi'(w)}{\psi(w) - z}\, w^n\, dw;$$

here

$$\psi(w) = \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots \quad (|w| > 1).$$

The uniqueness of the expansion is proved as in Faber' s theorem ([3]).

General case. For definiteness assume: $\infty \notin G$, $\infty \in D_0$. Let $w = \varphi_q(z)$ be a function conformally mapping the domain $G_q = (K_q, \Gamma_q)$ onto the annulus $1 < |w| < R_q$, with preservation of orientation. By Theorem 1 we have $\varphi_q(z) = \varphi_q^2(\varphi_q^1(z))$. Performing, if necessary, additional fractional-linear transformations, we choose the functions φ_q^1 and φ_q^2 in the following "canonical" way. For $q \neq 0$ the function $t_q = \varphi_q^1(z)$ maps one-to-one the domain $\Gamma_q(K_q)$ onto the domain $\Gamma_q^1(K_q^1)$, containing the point $t_q = \infty$, so that $\varphi_q^1(\infty) = \infty$; moreover $0 \in K_q^1(\Gamma_q^1)$. The function $w = \varphi_q^2(t_q)$ maps $K_q^1(\Gamma_q^1)$ one-to-one onto the circle $|w| > 1$ in such a way that $\varphi_q^2(0) = \infty$. The function $t_0 = \varphi_0^1(z)$ maps one-to-one the domain $\Gamma_0(K_0)$ onto the domain $\Gamma_0^1(K_0^1)$, which is the interior of the level line $\Gamma_0^1 = l_{R_0}^0$ of the bounded continuum $K_0^1 = \varphi_0^1(K_0)$.

Let $z = \psi_q(w)$, $z = \psi_q^1(t_q)$, $t_q = \psi_q^2(w)$ be the functions inverse, respectively, to $\varphi_q, \varphi_q^1, \varphi_q^2$. By $L_{\rho_q}^q$ $(1 < \rho_q < R_q)$ we denote the level lines of the domain $G_q = (K_q, \Gamma_q)$; $l_{\rho_q}^q = \psi_q^2(\{|w| = \rho_q\})$ $(\rho_q > 1)$. All contours are oriented in accordance with the mappings.

Definition of the basis functions

$$e_{qn}(z) = \varphi_{qn}^2(\varphi_q^1(z)) \qquad (q = 0, 1, \dots;\ n = 1, 2, \dots),$$

where

$$\varphi_{0n}^2(t_0) = \frac{1}{2\pi i} \int_{l_{\rho_0}^0} \frac{[\varphi_0^2(\tau_0)]^n}{\tau_0 - t_0}\, d\tau_0 = \alpha_n^{(n)} t_0^n + \alpha_{n-1}^{(n)} t_0^{n-1} + \cdots + \alpha_0^{(n)},$$

$$\varphi_{qn}^2(t_q) = \frac{1}{2\pi i} \int_{l_{\rho_q}^q} \frac{[\varphi_q^2(\tau_q)]^n}{\tau_q - t_q}\, d\tau_q = \frac{\beta_{q,n}^{(n)}}{t_q^n} + \frac{\beta_{q,n-1}^{(n)}}{t_q^{n-1}} + \cdots + \frac{\beta_{q,1}^{(n)}}{t_q},$$

or

$$e_{qn}(z) = \frac{1}{2\pi i} \int_{L_{\rho_q}^q} \frac{[\varphi_q(\zeta)]^n}{\varphi_q^1(\zeta) - \varphi_q^1(z)} \frac{d}{d\zeta} \varphi_q^1(\zeta)\, d\zeta, \qquad z \in L_{\rho_q}^q(K_q).$$

Formulas for the coefficients

$$a_{qn} = \frac{1}{2\pi i}\int_{|w|=\rho_q} f_q(\psi_q(w))\frac{1}{w^{n+1}}\,dw, \qquad \text{where} \quad f_q(z) = \frac{1}{2\pi i}\int_{L^q_{\rho_q}} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

The functions $f_q(\psi_q^1(t_q))$ are regular, respectively, in the domains $T_q^1(K_q^1)$, and for $q \neq 0$, $f_q(\psi_q^1(\infty)) = \infty$. To the function $f_0(\psi_0^1(t_0))$ we apply Faber's theorem, and to the remaining ones, the lemma. Taking into account Cauchy' s formula $f(z) = \sum_q f_q(z)$, we obtain the basic expansion. To prove uniqueness, multiply by $\frac{1}{2\pi i}\frac{1}{z-\zeta}$ and integrate with respect to $z \in L^q_{\rho_q}$. Properties 1°–4° can be derived from the explicit formulas.

4. Theorem 3. Let a continuum K and a simply connected domain G containing it be given. Let $e_0(z) \equiv 1,\ e_1(z), \dots, e_n(z), \dots$ be the corresponding principal basis. In order that a function $f(z)$, defined on K, be analytic in the domain G, it is necessary and sufficient that for every $\delta > 0$ and every natural n there exist a polynomial $\sum_{k=0}^n a_k e_k(z)$ such that

$$\left| f(z) - \sum_{k=0}^{n} a_k e_k(z) \right| \le C(\delta)\left(\frac{1+\delta}{R}\right)^n, \qquad z \in K$$

(the same can be said about polynomials $\sum_{k=0}^n b_k[\varphi^1(z)]^k$). Here $C(\delta)$ depends only on $\sup_{z\in G}|f(z)|$, if the function is bounded in G.

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References Cited

[1] S. N. Bernstein, *Communications of the Kharkov Mathematical Society*, ser. 2, **13** (1912). [2] G. Faber, *Math. Ann.*, **57**, 389 (1903). [3] A. I. Markushevich, *Theory of Analytic Functions*, 1950.

Note: Figure translations are in progress. See original paper for figures.