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Abstract

Full Text

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On Conformal Mappings

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Let a continuous function $f(z)$ be given in a domain D of the complex z-plane. We shall say that the mapping $w = f(z)$ **preserves angles** at a point $z \in D$ if the limit exists

$$\lim_{h \to 0} \operatorname{Arg} \frac{f(z+h) - f(z)}{h}. \tag{1}$$

Here values of the argument and their limits that differ by 2π must be regarded as identical*.

Theorem 1. *If a continuous mapping $w = f(z)$ preserves angles at every point of the domain D, except possibly for at most a countable set of them, then the function $f(z)$ is analytic inside D.*

This theorem was proved by D. E. Men'shov (1) under the additional assumption that the mapping $w = f(z)$ is single-valued.

Let again $f(z)$ be a continuous function in the domain D. We shall say that the mapping $w = f(z)$ **has constant stretching** at a point $z \in D$ if there exists the (finite or infinite) limit

$$\rho(z) = \lim_{h \to 0} \left| \frac{f(z+h) - f(z)}{h} \right|. \tag{2}$$

The following theorem holds, generalizing the well-known theorem of Bohr (2):

Theorem 2. *If a continuous mapping $w = f(z)$ is single-valued in the domain D and has constant stretching at each of its points, except possibly for at most a countable set of them, then inside D either the function $f(z)$ itself or its conjugate $\overline{f(z)}$ is analytic.*

This theorem can also be extended to the case of arbitrary noncontinuous mappings, if the property of preserving orientation at a point is defined in a suitable way.

Namely, let $w = f(z)$ be a continuous mapping of the domain D; consider some point $z_0 \in D$ and its image $w_0 = f(z_0)$ in the w-plane. We shall call the point

z_0 a U**-point** of the mapping $w = f(z)$ if there exists a neighborhood $V(z_0)$ of it such that for every point $z' \in V(z_0)$, $z' \neq z_0$, we have $f(z') \neq f(z_0)$**. Take an arbitrary closed Jordan curve $\lambda \subset V(z_0)$ enclosing the U-point z_0; it is clear that the continuous curve $l = f(\lambda)$ does not pass through the point $w_0 = f(z_0)$, and when the point z traverses the curve λ, the point $w = f(z)$ describes the whole curve l. If now, for positive traversal by the point z of the closed curve λ, the expression $\arg(w - w_0) = \arg[f(z) - f(z_0)]$ receives a nonnegative increment, and this holds for all possible $\lambda \subset V(z_0)$, then we shall say, that at the point $z_0 \in D$ the mapping $w = f(z)$ is direct (or preserves orientation).

One can prove the following generalization of Theorem 2:

Theorem 3. If an arbitrary continuous mapping $w = f(z)$ has constant stretching at each point of the domain D, with the exception, possibly, of at most a countable set of such points, and at each U-point, if such points exist, is direct, then the function $f(z)$ is analytic inside D. Moreover, if U-points do not exist, then $f(z)$ is constant.

The proof of all these theorems is based on the notion of a set of uniqueness (in the sense of N. N. Luzin) (3).

For the general case of a mapping with constant stretching one can prove the following result:

Theorem 4. If a continuous mapping $w = f(z)$ has constant stretching at each point of the domain D, with the exception, possibly, of at most a countable set of such points, and, moreover, $\rho(z)$ (2) can be equal to zero only on a set of points of category I (in D), then there exists a set O, open and everywhere dense in D, in each component of which either the function $f(z)$ itself or its conjugate $\overline{f(z)}$ is analytic.

An example of such a mapping is given by the function already indicated by Bohr (2):

$$f(z) = \begin{cases} z, & \text{for } \operatorname{Im} z > 0, \\ z, & \text{for } \operatorname{Im} z \leq 0. \end{cases}$$

Some generalizations of the results of D. E. Menshov and H. Bohr were formulated by Kuramochi (4) for the case of univalent mappings $w = f(z)$ under a

* We assume that the values of Arg in expression (1) depend continuously on h; note also that expression (1) has meaning only in the case when $f(z+h) \neq f(z)$ for sufficiently small $|h|$.

** Note that if the mapping $w = f(z)$ preserves angles at a point $z_0 \in D$, then z_0 is a U-point, which follows from the very meaning of expression (1).

certain additional restriction. In the hypotheses of the theorems of the present note only the continuity of the mapping $w = f(z)$ is assumed.

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CITED LITERATURE

[1] D. Menshov, *Math. Ann.*, **95**, 641 (1926).
[2] H. Bohr, *Math. Zs.*, **1** (1918).
[3] Yu. Yu. Trokhimchuk, *Uspekhi Mat. Nauk*, vol. 5, 215 (1956).
[4] Z. Kuramochi, *Osaka Math. J.*, **3**, No. 1, 21 (1951).

Note: Figure translations are in progress. See original paper for figures.