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Abstract

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ON THE EMBEDDING OF NORMAL SPACES IN BICOMPACTA OF THE SAME WEIGHT AND THE SAME DIMENSION

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In the present paper a positive answer is given to the following question posed by P. S. Aleksandrov: **can every normal space be topologically embedded in a bicompactum of the same weight and the same dimension?*** For spaces with a countable base this was proved long ago by Hurewicz ([3]) and Tumarkin. For spaces that are zero-dimensional in the inductive sense, this was proved by N. B. Vedenisov ([4]). For spaces of positive inductive dimension this is no longer true, as is shown by an example of Yu. M. Smirnov ([5]).

In application to arbitrary normal spaces, the following theorem, proved by Hurewicz ([3]) for spaces with a countable base, also turns out to be valid: *for any countable system of closed sets A_i of a space R, there exists a bicompact extension R^* of the same weight such that* $\dim R^*[A_i] = \dim A_i$ *for all i.*

Let R be a normal n-dimensional space of weight τ. The construction of a bicompact extension R^* of the space R of dimension n and weight τ is based on the theory of proximity spaces and uniform spaces developed by Yu. M. Smirnov. According to this theory ([1]), there exists, and moreover only one, bicompact extension λR of the space R inducing on it a given proximity R_λ: *two sets are close if and only if their closures in the bicompact extension λR intersect.* One of the ways of specifying a proximity on a topological space is the construction of a uniform structure Σ_λ from finite open coverings, which turn out, in the proximity R_λ generated by them, to be the so-called δ-coverings (([1]), p. 559) and extend to open coverings of the bicompact extension λR.

In order that the bicompact extension λR (corresponding to the proximity thus constructed) have weight τ and dimension $\leq n$, it is necessary and sufficient that the uniform structure Σ_λ satisfy the following conditions:

A. In Σ_λ there is a cofinal part of cardinality not exceeding τ.

B. In Σ_λ there is a cofinal part consisting of coverings of multiplicity not exceeding $n + 1$.**

Lemma 1. *On the space R there exists a uniform structure*** Σ_1, possessing the following properties:*

* By dimension here and throughout is meant the dimension dim, defined by means of finite open coverings. By inductive dimension is meant only the small inductive dimension ind of Urysohn, in the definition of which induction is carried out over points.

** This means that the proximity space R_λ defined by the structure Σ_λ has dimension $\leq n$ in the sense of Yu. M. Smirnov ([2]).

*** Here and below, by a uniform structure is also meant any cofinal part of a uniform structure in the sense of Yu. M. Smirnov (([2]), p. 563), i.e. one satisfying only conditions C2 and C3 of the system of coverings.

a) the structure Σ_1 has cardinality τ;

b) for every cover α from Σ_1 there are only finitely many covers β from Σ_1 such that $\beta < \alpha$*.

Proof. By the well-known theorem of A. N. Tikhonov, the space R can be topologically embedded in the product $I^\tau = \prod_\lambda I_\lambda$ of intervals I_λ, taken in number τ. In the structure $\widetilde{\Sigma}$ of this bicompactum I^τ, consider the following cofinal part $\widetilde{\Sigma}_1$: on each interval I_λ fix a countable uniform structure of covers, successively star-refined in one another. **These structures satisfy condition b). Let $\{\alpha^i\}$, $i = 1, \dots, s$, be a system of covers $\alpha^i = \{\Gamma^i_j\}$, taken one from each of the fixed structures of the intervals I_{λ_i}, $i = 1, \dots, s$. By the** product** $\prod \alpha^i$ of the covers α^i we shall mean the cover $\tilde{\alpha}$ of the bicompactum I^τ by sets of the form $\prod_\lambda \Gamma_\lambda$, where

$$\Gamma_\lambda = \Gamma^i_j, \quad \text{if } \lambda = \lambda_i \text{ for } i = 1, \dots, s, \quad \text{and } \Gamma_\lambda = I_\lambda \text{ in the remaining cases.}$$

The covers $\tilde{\alpha}$ of the bicompactum I^τ constitute the required cofinal part $\widetilde{\Sigma}_1$ of the uniform structure $\widetilde{\Sigma}$, satisfying, obviously, conditions a) and b) with the usual ordering by refinement.

Now let Σ_1 be the uniform structure of the space R cut out by the structure $\widetilde{\Sigma}_1$. For covers α and β from Σ_1 we shall consider that $\alpha < \beta$ if and only if, for the covers $\tilde{\alpha}$ and $\tilde{\beta}$ cutting them out, the cover $\tilde{\alpha}$ is refined into $\tilde{\beta}$; moreover, covers cut out by different covers of the bicompactum will also be considered different. It is not hard to see that the uniform structure Σ_1 satisfies conditions a) and b), which was required to prove.

Lemma 2. There exist a cofinal part Σ' of the uniform structure Σ_1 and a system Σ of covers of the space R such that between Σ' and Σ there is a one-to-one correspondence satisfying the following conditions:

1) the covers from Σ have multiplicity $\leqslant n + 1$;

2) if a from Σ and a' from Σ' correspond to each other, then a is refined into a';

3) if $\alpha' < \beta'$, $\alpha', \beta' \in \Sigma'$, in the sense of the order of the uniform structure Σ_1, then for the corresponding covers from Σ we have $\alpha < *\beta$.

Proof. Consider the set S of systems π of pairs of covers (α', α), where $\alpha' \in \Sigma_1$, α has multiplicity $\leqslant n+1$; α is refined into α'; if $(\alpha', \alpha), (\beta', \beta) \in \pi$ and $\alpha' < \beta'$, then $\alpha < *\beta$. Obviously, the set S is nonempty. It is partially ordered: $\pi_1 < \pi_2$ if $\pi_1 \subset \pi_2$. Every ordered subset (chain) M of the set S is bounded above by the union of the systems belonging to it. Therefore, by the Hausdorff–Zorn lemma, every system of pairs from S is contained in some maximal system of pairs. Let π be a maximal system of pairs of the set S. Denote by Σ' the part of the uniform structure Σ_1 through which the covers α' of the system π run, and by Σ the corresponding collection of covers α. The systems Σ' and Σ of covers are the desired ones. Indeed, conditions 1), 2), 3) are fulfilled, and it remains only to prove that the system Σ' is cofinal in the uniform structure Σ_1. But assuming that this is not so, in the structure Σ_1 we would find a cover α'_0 after which there follows none of the covers of the system Σ'. By property b) (see Lemma 1), in the system Σ' there are only finitely many covers $\alpha'_1, \dots, \alpha'_k$ preceding the cover α'_0. Let α_0 be a cover of multiplicity $\leqslant n+1$, star-refined into

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* The relation $\beta < \alpha$ here means that the cover β is not only refined into α, but also follows a in the sense that will be clear from the proof. By a cover we shall understand everywhere only finite open covers.

** A cover β is **star-refined** into a cover α ($\beta < *\alpha$) if for every point x of the space R, the union of all elements of the cover β containing the point x is contained in some element of the cover α.

into the cover $\alpha'_0 \wedge \alpha_1 \wedge \dots \wedge \alpha^*_k$, where α_i are covers of the system Σ corresponding to the covers α'_i, $i = 1, \dots, k$. Then the system of pairs obtained by adjoining the pair (α'_0, α_0) to the maximal system π must belong to the set S, contrary to the maximality of the system π. The lemma is proved.

Lemma 3. *The system Σ is a uniform structure of the space R and satisfies conditions A and B.*

Proof. By the construction of the system Σ, for every cover α from Σ there exists a cover $\beta \in \Sigma$ star-refined in α. For any covers α and β there exists a cover γ such that $\gamma > \alpha \wedge \beta$ (i.e. $\gamma > \alpha$ and $\gamma > \beta$). Indeed, take the corresponding covers α' and β' from Σ', and choose in the structure Σ' a cover γ' such that $\gamma' > \alpha' \wedge \beta'$. We then have $\gamma > \alpha$ and $\gamma > \beta$. Note that the uniform structure Σ is compatible with the topology of the space R, since for any point $x \in R$ and any of its neighborhoods Ox there is a cover $\alpha \in \Sigma$ such that $O_\alpha x \subseteq Ox$**. Lemma 3 is proved. Hence the following theorem is also proved:

Theorem 1. *Every normal space has a bicompact extension of the same weight and the same dimension.*

Remark. The bicompact extension corresponding to the structure Σ has dimension $\leq n$. To obtain an extension of dimension n, one must take not an arbitrary maximal system π of pairs of covers, but such a maximal system which contains a fixed pair of covers (α', α), where α is a cover of multiplicity $n+1$ into which no cover of smaller multiplicity can be inscribed.

Theorem 2. *For any countable system of closed sets A_k of a normal space R, there exists a bicompact extension R^* of the space R of the same weight and such that* $\dim R^*[A_k] = \dim A_k$ *for every k.*

Proof of this theorem proceeds analogously to the preceding proof. Therefore here we shall indicate only the main points.

Let $\{A_k\}$ be a countable system of closed sets of the space R. We shall say that a cover $\gamma = \{\Gamma_j\}$ has rank N (where N is a natural number or zero) if on each of the sets A_k, where $k \leq N$, it cuts out a cover $\{\Gamma_j \cap A_k\}$ of multiplicity $\leq \dim A_k + 1$.

Lemma 4. *For any N, into every cover of the space R one can inscribe a cover of rank N.*

Proof. For $N = 0$ the assertion of the lemma is valid. Suppose that for $N = k-1$ the assertion of the lemma is true, and prove it for $N = k$. Then it is enough to show that into every cover $\gamma = \{\Gamma_j\}$ of rank $k-1$ one can inscribe a cover of rank k. For this, into the cover $\{\Gamma_j \cap A_k\}$ of the set A_k we inscribe combinatorially a closed cover $\{\Phi_j\}$ of multiplicity $\leq \dim A_k + 1$ so that $\Phi_j \subseteq \Gamma_j \cap A_k$. For the system of closed sets $\{\Phi_j\}$ of the space R there exists a similar system of neighborhoods $O\Phi_j$ such that $O\Phi_j \subseteq \Gamma_j$. Then the system of sets $O\Phi_j \cup (\Gamma_j \setminus A_k)$ will be the required one. The lemma is proved.

For the proof of Theorem 2 it is enough to construct a uniform structure Σ of the space R satisfying the following conditions:

A′. In Σ there is a cofinal part of cardinality $\leq \tau$.

B′. For every number N, covers of rank N constitute a cofinal part of the structure Σ.

According to the preceding proof there exists a uniform structure Σ_1 of the space R such that for each cover α from Σ_1 in it

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$*$ $\alpha \wedge \beta$ is the cover consisting of all possible intersections $A \cap B$, where $A \in \alpha$, $B \in \beta$.

$**$ $O_\alpha x$ is the sum of all elements of the cover α containing the point x. For this it is enough to find a cover $\alpha' \in \Sigma'$ such that $O_{\alpha'} x \subseteq Ox$, and take the cover α

corresponding to the cover α'. Of course, one must also keep in mind that the covers α are open.

there is only a finite number of preceding (in the same special sense) coverings. We shall call this number the *number* of the covering α. In any cofinal part of the uniform structure Σ_1', the coverings with numbers $\geq n$ obviously form a cofinal part.

Lemma $2'$**.** *There exist a cofinal part* Σ' *of the uniform structure* Σ_1' *and a system of coverings* Σ *of the space* R *such that there is a one-to-one correspondence between* Σ' *and* Σ *satisfying the following conditions:*

1') If a covering α' *from* Σ' *has number* N*, then the corresponding covering* α *from* Σ *has order* N*;*

2') If $\alpha' \in \Sigma'$*, then the corresponding covering* α *from* Σ *is inscribed in* α'*;*

3') If $\beta' > \alpha'$ *(in the sense of the order of the structure* Σ_1*), then* $\beta^* > \alpha$*.*

Lemma $3'$**.** *The system* Σ *is a uniform structure of the space* R *and satisfies conditions A' and B' .*

The proofs of these lemmas proceed analogously to the proofs of Lemmas 2 and 3.

Remark. The bicompact extension R^* corresponding to the constructed uniform structure Σ has the property that $\dim R[A_n] \leq \dim A_n$ for every n. To construct a uniform structure Σ generating an extension R^* satisfying the conditions of Theorem 2, instead of an arbitrary maximal system π of pairs of coverings one should take a maximal system containing a sequence of pairs (α_n', α_n) of coverings α_n', α_n, satisfying conditions $1'$, $2'$, and $3'$, and such that in the covering α_n one cannot inscribe coverings of the set A_n of multiplicity less than $\dim A_n + 1$; and such a sequence of pairs can always be constructed. Theorem 2 is proved.

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Note: Figure translations are in progress. See original paper for figures.