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Abstract

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ASYMPTOTICS OF SOLUTIONS OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR EQUATIONS WITH A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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1. Statement of the problem. Consider the equation

$$\mu \frac{d^2 y}{dt^2} = -A(t, y)\frac{dy}{dt} + B(t, y), \tag{1}$$

where $\mu \geqslant 0$ is a small parameter. A number of works are devoted to the study of the solution of equation (1) and of equations of a more general form under various boundary conditions (see, for example, ([1])).

Let the boundary conditions be given as

$$\text{I.} \quad y'(0) = 0, \quad y(1) = 0; \tag{2'}$$

$$\text{II.} \quad y(0) = 0, \quad y(1) = 0. \tag{2''}$$

We shall denote the corresponding solutions of equation (1) by $y_{\mathrm{I}}(t, \mu)$, $y_{\mathrm{II}}(t, \mu)$.

Putting $\mu = 0$ in (1), we obtain the degenerate equation

$$A(t, u)\frac{dy}{dt} = B(t, u), \tag{3}$$

whose solution we define by the condition

$$u(1) = 0. \tag{4}$$

In ([1]), under certain restrictions on the right-hand side of (1) (see below), it is shown that

$$\lim_{\mu\to 0} y_{\rm I}(t,\mu) = u(t), \quad 0 \leqslant t \leqslant 1; \tag{5'}$$

$$\lim_{\mu\to 0} y_{\rm II}(t,\mu) = u(t), \quad 0 < t \leqslant 1. \tag{5''}$$

Thus, for sufficiently small values of μ, $u(t)$ can serve as an approximate solution both for problem I and for problem II. In case I the approximation is uniform with respect to t on the entire interval $0 \leqslant t \leqslant 1$; in case II the approximation is not uniform on $0 \leqslant t \leqslant 1$ and loses its meaning in a neighborhood of $t = 0$. It is natural to pose the question of constructing a uniform approximation in case II and of constructing approximations of higher order both for case II and for case I.

2. Consider the following auxiliary problem with initial conditions. Rewrite (1) in the form of the system

$$\mu\frac{dz}{dt} = -A(t,y)z + B(t,y), \quad \frac{dy}{dt} = z. \tag{6}$$

We shall require that $A(t,y)$, $B(t,y)$ possess continuous partial derivatives up to and including the second order in the domain $0 \leq t \leq 1$, $|y| < d$, and that in this domain the condition $-A(t,y) < -x < 0$ ($x = \text{const}$) be satisfied. We prescribe for (6) the initial conditions

$$z\big|_{t=0} = z_0^0, \qquad y\big|_{t=0} = y_0^0 + \mu y_1^0. \tag{7}$$

The results and methods of papers ($^{2-4}$) make it possible to justify the following rule for constructing an asymptotic formula for the solution $z(t,\mu)$, $y(t,\mu)$ of system (6), under the initial conditions (7), with accuracy $O(\mu^2)$ on $0 \leq t \leq 1$. Introduce a new independent variable $\tau = t/\mu$, and rewrite (6) in the form

$$\frac{dz}{d\tau} = -A(\tau\mu,y)z + B(\tau\mu,y), \qquad \frac{dy}{d\tau} = \mu z \tag{8}$$

and seek the solution in the form of formal series

$$z = z_0(\tau) + \mu z_1(\tau) + \cdots, \qquad y = y_0(\tau) + \mu y_1(\tau) + \cdots. \tag{9}$$

Then

$$\frac{dz_0}{d\tau} = -A(0,y_0)z_0 + B(0,y_0), \qquad \frac{dy_0}{d\tau} = 0,$$

$$z_0\big|_{\tau=0} = z_0^0, \qquad y_0\big|_{\tau=0} = y_0^0; \tag{10'}$$

$$\frac{dz_1}{d\tau} = -A(0, y_0)z_1 + [-A_y(0, y_0)z_0 + B_y(0, y_0)]y_1 +$$

$$+[-A_t(0, y_0)z_0 + B_t(0, y_0)]\tau, \qquad \frac{dy_1}{d\tau} = z_0, \tag{10''}$$

$$z_1\big|_{\tau=0} = 0, \qquad y_1\big|_{\tau=0} = y_1^0.$$

Let us write the degenerate system of equations corresponding to (6), i.e. obtained from (6) if one formally sets $\mu = 0$:

$$A(t, \bar{y})\bar{z} = B(t, \bar{y}), \tag{11}$$

and consider its solution under the initial condition

$$\bar{y}\big|_{t=0} = y_0^0. \tag{12}$$

Differentiate (6) with respect to μ and again carry out the degeneration. We obtain the system

$$\frac{d}{dt}\bar{z} = -A(t, \bar{y})\bar{z}_\mu + [-A_y(t, \bar{y})\bar{z} + B_y(t, \bar{y})]\bar{y}_\mu, \qquad \frac{d}{dt}\bar{y}_\mu = \bar{z}_\mu \tag{13}$$

and find its solution satisfying the initial conditions (4)

$$\bar{y}_\mu\big|_{t=0} = \bar{y}_\mu(0) = y_1^0 - \int_0^\infty \tau z_0'(\tau)\, d\tau. \tag{14}$$

Form the expressions

$$Z_0 = \bar{z} + z_0 - \bar{z}(0), \qquad Y_0 = \bar{y} + y_0 - \bar{y}(0) = \bar{y} + y_0^0 - y_0^0 = \bar{y}; \tag{15'}$$

$$Z_1 = \bar{z} + \mu\bar{z}_\mu + z_0 + \mu z_1 - (\bar{z}(0) + t\bar{z}'(0) + \mu z_\mu(0)),$$

$$Y_1 = \bar{y} + \mu\bar{y}_\mu + \mu y_1 - (t\bar{y}'(0) + \mu\bar{y}_\mu(0)). \tag{15''}$$

Using the methods of ([4]), one can prove that

$$|z - Z_0| < c\mu, \qquad |y - Y_0| < c\mu, \tag{16'}$$

$$|z - Z_1| < c\mu^2, \qquad |y - Y_1| < c\mu^2, \tag{16''}$$

where c is a constant independent of μ and t $(0 \le t \le 1)$, for sufficiently small $\mu \le \mu_0$.

The scheme for obtaining asymptotic formulas can be developed further and approximations of a higher degree of accuracy μ^n can be constructed both for the initial conditions (7) and for a broader class

$$z\big|_{t=0} = z_0^0 + \mu z_1^0 + \cdots + \mu^k z_k^0, \qquad y\big|_{t=0} = y_0^0 + \mu y_1^0 + \cdots + \mu^k y_k^0.$$

We now prescribe the initial conditions for (6) in the form

$$z\big|_{t=0} = \frac{z_{-1}^0}{\mu}, \qquad y\big|_{t=1} = y_0^0 \tag{17}$$

and, as in the case (7), we shall construct the solution (8) in the form of formal series

$$z = \frac{z_{-1}(\tau)}{\mu} + z_0(\tau) + \mu z_1(\tau) + \cdots, \qquad y = y_0(\tau) + \mu y_1(\tau) + \cdots. \tag{18}$$

Here

$$\frac{dz_{-1}}{d\tau} = -A(0, y_0) z_{-1}, \qquad \frac{dy_0}{d\tau} = z_{-1},$$

$$z_{-1}\big|_{\tau=0} = z_{-1}^0, \qquad y_0\big|_{\tau=0} = y_0^0. \tag{19}$$

We determine the solution $\bar{z}$, $\bar{y}$ of the degenerate system (11) by the condition*

$$\bar{y}\big|_{t=0} = \bar{y}(0) = y_0^0 + \int_0^\infty z_{-1}(\tau)\, d\tau. \tag{20}$$

By a method analogous to that by which the inequalities (16) were proved, one can prove for the present case that

$$Y_0 = y_0 + \bar{y} - \bar{y}(0) \tag{21}$$

is a uniform approximation to the solution $y(t, \mu)$ such that

$$|y - Y_0| < c\mu, \tag{22}$$

where c is a constant independent of μ and t $(0 \le t \le 1)$, for sufficiently small $\mu \le \mu_0$.

3. The idea of applying the indicated general scheme to the solution of boundary-value problems is as follows: to choose the parameters z^0_{-1}, y^0_0, etc. in the initial conditions (7) or (17) in such a way as to satisfy the prescribed boundary conditions.

Consider boundary-value problem I. Put $z^0_0 = 0$ in the initial conditions (7). Determine y^0_0 from the requirement $y^0_0 = \bar{y}\big|_{t=0}$, where $\bar{y}$ is the solution of the degenerate system (11) satisfying the condition $\bar{y}(1) = 0$ ($\bar{y}$, obviously, coincides with u in (3)), and determine y^0_1 from the requirement

$$y^0_1 - \int_0^\infty \tau z'_0(\tau)\, d\tau = \bar{y}_\mu\big|_{t=0},$$

where y_μ is the solution of the degenerate system (13) satisfying the condition $y_\mu\big|_{t=1} = 0$. The solution of system (6) obtained with such a choice of parameters in the initial conditions (7) will have the property

$$y\big|_{t=1} = \left(\bar{y} + \mu\bar{y}_\mu + \frac{\mu^2}{2}\bar{y}^*_{\mu\mu}\right)_{t=1} = \frac{\mu^2}{2}\bar{y}^*_{\mu\mu}\big|_{t=1} = O(\mu^2), \qquad z\big|_{t=0} = 0.$$

At the same time the solution

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* $\bar{y}(0)$ can also be determined from the equation $\displaystyle\int_{y^0_0}^{\bar{y}(0)} A(0, y)\, dy = z^0_{-1}$.

of boundary-value problem I has the property $y|_{t=1} = 0$, $z|_{t=0} = 0$. By the methods of (3) it is not difficult to obtain that the difference between the indicated solution of the problem with initial conditions and the solution of boundary-value problem I is of order $O(\mu^2)$, uniformly with respect to t on the whole interval $0 \le t \le 1$. And since formulas (15″) constitute, uniformly with accuracy $O(\mu^2)$, an approximation to the solution of the problem with initial conditions, the following basic assertion is valid:

The expressions

$$Y_1 = \bar{y} + \mu\bar{y}_\mu + \mu y_1 - t\bar{y}'(0) - \mu\bar{y}_\mu(0),$$

$$Z_1 = \bar{z} + \mu\bar{z}_\mu + z_0 + \mu z_1 - \bar{z}(0) - t\bar{z}'(0) - \mu\bar{z}_\mu(0) \tag{23}$$

(where $\bar{y}, \bar{z}$ are determined by system (11) and the condition $\bar{y}|_{t=1} = 0$; $\bar{y}_\mu, \bar{z}_\mu$ are determined by system (13) and the condition $\bar{y}_\mu|_{t=1} = 0$; z_0 is determined by system (10′) with initial conditions $z_0|_{t=0} = 0$, $y_0|_{t=0} = \bar{y}(0)$; y_1, z_1 are determined by system (10″) with initial conditions

$$z_1|_{t=0} = 0, \qquad y_1|_{t=0} = \bar{y}_\mu(0) + \int_0^\infty \tau z_0'(\tau)\, d\tau$$

) are, with accuracy $O(\mu^2)$, an approximate solution of boundary-value problem I for equation (1) under condition (2′), so that

$$|\, y(t,\mu) - Y_1 \,| < c\mu^2, \qquad |\, y'(t,\mu) - Z_1 \,| < c\mu^2$$

on the whole interval $0 \le t \le 1$; c is a constant independent of μ for sufficiently small $\mu \le \mu_0$.

We note that, prescribing for (6) the initial conditions $z|_{t=0} = 0$, $y|_{t=0} = y_0^0 + \mu y_0^1 + \cdots$, one can similarly obtain the following approximations for the solution of boundary-value problem I.

Similarly, one can consider boundary-value problem II and assert:

The expression

$$Y_0 = \bar{y} + y_0 - \bar{y}(0) \tag{24}$$

(where $\bar{y}$ is determined from system (11) with the initial condition $\bar{y}|_{t=1} = 0$; y_0 is determined from (19) with the initial condition

$$z_{-1}|_{t=0} = z_{-1}^0 = \int_0^{\bar{y}(0)} A(0,y)\, dy, \qquad y_0|_{t=0} = 0$$

) is, with accuracy $O(\mu)$, an approximate solution of boundary-value problem II for equation (1), so that

$$|\, y(t,\mu) - Y_0 \,| < c\mu$$

on the whole interval $0 \le t \le 1$; c is a constant independent of μ for sufficiently small values $\mu \le \mu_0$.

As in case I, one can develop this scheme and write approximations of higher order.

In conclusion we note that formulas (23), (24) are applicable to the more general boundary-value problem for equation (1)

$$y(0) + \alpha y'(0) = 0, \qquad y(1) + \beta y'(1) = 0 \tag{25}$$

with an obvious modification of the initial conditions for the functions entering into (23) in the case $\alpha \neq 0$, and into (24) in the case $\alpha = 0$.

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Note: Figure translations are in progress. See original paper for figures.