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Abstract

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INTEGRAL REPRESENTATIONS OF FUNCTIONS OF TWO COMPLEX VARIABLES

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In papers ($^{2-4}$) we established that if a function $F(w,z)$ is regular in a domain $D \ni (0,0)$ of the space of two complex variables w, z, bounded by the hypersurface

$$|w| = r_1(\tau), \qquad |z| = r_2(\tau), \qquad 0 \leq \tau \leq 1,$$

where $r_1(\tau)$ is a continuous function on the segment $[0,1]$ and

$$r_1(0) = 0, \qquad 0 < r_1'(\tau) \leq \frac{r_1(\tau)}{\tau}, \qquad r_1(1) < \infty, \qquad 0 < \tau \leq 1; \tag{1}$$

$$r_2(\tau) = \exp\left[-\int \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right], \tag{2}$$

and is continuous in the closed domain $\overline{D}$, then for a point $(w,z) \in D$

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_C \frac{\zeta F[r_1(\tau)\zeta, r_2(\tau)\eta]}{(\zeta-u)^2}\, d\zeta. \tag{3}$$

If, however, $F(w,z)$ is a function of class α, i.e. $F(w,z)$ is analytic in D and the functions $F(w,z)$, $F_w'(w,z)$, $F_z'(w,z)$ are continuous in the closed domain $\overline{D}$, then for a point $(w,z) \in D$

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_C \frac{\Phi(r_1(\tau)\zeta, r_2(\tau)\eta)}{\zeta-u}\, d\zeta. \tag{4}$$

Here C is the circle $|\zeta| = 1$, $\eta = \zeta e^{-it}$, $u = \tau\frac{w}{r_1(\tau)} + (1+\tau)\frac{z}{r_2(\tau)}e^{it}$,

$$\Phi(w,z) = L_1[F] \equiv F(w,z) + wF_w'(w,z) + zF_z'(w,z).$$

Formula (3) expresses the value of $F(w,z)$ inside the domain D in terms of its values on the boundary of this domain; formula (4) establishes a property inherent in functions of class α: the values of the function $F(w,z)$ in the domain D are determined by the behavior of the linear differential operator $L_1[F]$ on the boundary of the domain $\overline{D}$.

Formulas (2), (1), defining the functions $r_1(\tau), r_2(\tau)$, express the following requirement: in the "absolute quadrant" (in which the coordinates are $|w|, |z|$) the curve determined by the equations

$$|w| = r_1(\tau), \qquad |z| = r_2(\tau), \qquad 0 \le \tau \le 1,$$

is the envelope of the family of straight lines given by the equation

$$\tau \frac{|w|}{r_1(\tau)} + (1-\tau)\frac{|z|}{r_2(\tau)} = 1, \qquad 0 < \tau < 1,$$

and is situated under the envelope ([2,3]).

The question arises how broad is the class of domains, determined by conditions (1) and (2), for which the integral representations (3) and (4) hold.

Along with this class, let us consider the class of domains determined by conditions (2) and

$$r_1(0) = 0, \qquad 0 < r_1'(\tau) \gtrless \frac{r_1(\tau)}{\tau}, \qquad r_1(1) < \infty, \qquad 0 < \tau \le 1 \qquad (1')$$

(the function $r_1'(\tau) - \dfrac{r_1(\tau)}{\tau}$ either changes sign on the interval $0 < \tau < 1$ or is positive).

Concerning these domains, it turns out that the following theorem holds:

Theorem. *The class of domains whose boundaries are defined either by conditions (1), (2) or by conditions (1′), (2) coincides with the class of bicircular domains whose boundaries* $\Phi_1(w,\bar{w},z,\bar{z}) = 0$ *are twice continuously differentiable and analytically convex from the outside* ($L(\Phi_1) > 0$, $L(\Phi_1)$ *is the Levi determinant). For domains determined by conditions (1′), (2), the integral representations hold*

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_C \frac{\zeta F[r_1(\tau)\zeta^n,\ r_2(\tau)\eta^n]}{(\zeta-u)^2}\, d\zeta; \qquad (3')$$

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_C \frac{\Phi[r_1(\tau)\zeta^n,\ r_2(\tau)\eta^n]}{\zeta-u}\, d\zeta, \qquad (4')$$

where

$$u = \tau \left(\frac{w}{r_1(\tau)} \right)^{1/n} + (1-\tau) \left(\frac{z}{r_2(\tau)} \right)^{1/n} e^{it}, \qquad n = (\mu), \qquad \mu = \sup_{0<\tau<1} \frac{d \ln r_1(\tau)}{d \ln \tau},$$

$$\Phi(w,z) = L_n[F] \equiv F(w,z) + nwF'_w(w,z) + nzF'_z(w,z),$$

(μ) is the least integer not less than μ.

Proof. First we shall establish that there exists a double power series for which the quantities $r_1 = r_1(\tau)$, $r_2 = r_2(\tau)$, determined by conditions (1), (2) or (1′), (2), are conjugate radii of convergence.

Indeed, by condition (1) or (1′), $r_1 = r_1(\tau)$ is a positive, continuous, monotonically increasing function on the interval $(0,1)$. Therefore the inverse function $\tau = \psi(r_1)$ is continuous and monotonically increasing on the interval $(0, R_1)$, where $R_1 = r_1(1)$. By virtue of condition (2) and (1) or (1′),

$$r'_2(\tau) = -\frac{\tau}{1-\tau} \frac{r'_1(\tau)}{r_1(\tau)} r_2(\tau) < 0$$

on the interval $0 < \tau < 1$. Consequently, $r_2 = r_2(\tau)$ on the interval $(0,1)$ is a positive, continuous, monotonically decreasing function, and therefore

$$r_2 = r_2[\psi(r_1)] \equiv \Phi_2(r_1)$$

is a positive, continuous, monotonically decreasing function on the interval $(0, R_1)$. It is known ([2]) that

$$\lim_{r_1 \to R_1} \Phi_2(r_1) = \lim_{\tau \to 1} r_2(\tau) = 0.$$

It remains for us to make sure that the curve $r_2 = \Phi_2(r_1)$

has the property of "logarithmic convexity." From condition (2) we have

$$\ln r_2(\tau) = -\int \frac{\tau}{1-\tau} \, d \ln r_1(\tau)$$

or

$$\frac{d \ln r_2(\tau)}{d \ln r_1(\tau)} = -\frac{\tau}{1-\tau}.$$

Hence

$$\frac{d^2 \ln r_2(\tau)}{(d \ln r_1(\tau))^2} = -\frac{1}{(1-\tau)^2} \frac{r_1(\tau)}{r_1'(\tau)} < 0, \qquad 0 < \tau < 1,$$

i.e.

$$\frac{d^2 \ln \Phi_2(r_1)}{(d \ln r_1)^2} = -\frac{r_1 \psi'(r_1)}{[1-\psi(r_1)]^2} < 0, \qquad 0 < r_1 < R_1.$$

It follows from Hartogs' theorem (1) that there exists a double power series for which the quantities $r_1 = r_1(\tau)$, $r_2 = r_2(\tau)$ are conjugate radii of convergence.

Thus, the domain $D \ni (0,0)$ with boundary $|w| = r_1(\tau)$, $|z| = r_2(\tau)$, where $r_1(\tau)$, $r_2(\tau)$ are determined either by conditions (1), (2) or by conditions (1′), (2), is a complete bicircular domain whose boundary

$$\Phi_1(w, \bar{w}, z, \bar{z}) = 0$$

is twice differentiable and satisfies the condition $L(\Phi_1) > 0$.

Conversely, let the domain $D \ni (0,0)$ be any bounded, complete bicircular domain whose boundary

$$\Phi_1(w, \bar{w}, z, \bar{z}) \equiv \sqrt{z\bar{z}} - \Phi_2(\sqrt{w\bar{w}}) = 0,$$

$0 \le |w| \le R_1$, is twice continuously differentiable and satisfies the condition $L(\Phi_1) > 0$. We shall prove the existence of a (unique) function $\varphi(\tau)$, $0 \le \tau \le 1$, such that the boundary $|z| = \Phi_2(|w|)$ of the domain D is determined by the parametric equations

$$|w| = \varphi(\tau) \equiv r_1(\tau), \qquad |z| = \Phi_2[\varphi(\tau)] \equiv r_2(\tau), \qquad 0 \le \tau \le 1,$$

where $r_1(\tau)$ and $r_2(\tau)$ satisfy either conditions (1), (2) or conditions (1′), (2). The desired function $\varphi(\tau)$ must be a solution of the functional equation

$$\Phi_2[\varphi(\tau)] = \exp\left[-\int \frac{\tau}{1-\tau} \frac{\varphi'(\tau)}{\varphi(\tau)}\, d\tau\right]$$

or of the equation

$$\frac{\Phi_2'[\varphi(\tau)]\varphi'(\tau)}{\Phi_2[\varphi(\tau)]} = -\frac{\tau}{1-\tau} \frac{\varphi'(\tau)}{\varphi(\tau)}.$$

With the exception of the case $r_1(\tau) = \varphi(\tau) \equiv \text{const}$, the last equation takes the form

$$\frac{\Phi_2'[r_1(\tau)]\,r_1(\tau)}{\Phi_2[r_1(\tau)]} = -\frac{\tau}{1-\tau} \tag{5}$$

or the form

$$\frac{d\ln\Phi_2[\varphi(\tau)]}{d\ln\varphi(\tau)} \equiv \frac{d\ln r_2(\tau)}{d\ln r_1(\tau)} = -\frac{\tau}{1-\tau}. \tag{6}$$

Since $L(\Phi_1) > 0$, the function $r_2 = \Phi_2(r_1)$ satisfies the condition

$$\frac{d^2\ln r_2}{(d\ln r_1)^2} < 0.$$

Therefore the left-hand side of equation (6), as a function of $\ln\varphi(\tau)$, is a monotonically decreasing function. Consequently, there exists a (unique) solution of equation (6)

$$\ln\varphi(\tau) = \psi\left(-\frac{\tau}{1-\tau}\right),$$

where $\psi\left(-\frac{\tau}{1-\tau}\right)$ is a monotonically decreasing function of the argument $-\frac{\tau}{1-\tau}$, or, what is the same, a monotonically increasing function of τ. Therefore

$$\varphi(\tau) = \exp\psi\left(-\frac{\tau}{1-\tau}\right)$$

is a positive, monotonically increasing function of the variable τ, such that the functions $r_1(\tau) \equiv \varphi(\tau)$, $r_2(\tau) \equiv \Phi_2[\varphi(\tau)]$ satisfy condition (2).

It remains for us to establish that the function $\varphi(\tau) \equiv r_1(\tau)$ satisfies either condition (1) or condition (1′). From formula (5) it is clear that $r_1(0) = 0$. Indeed, otherwise we would have $r_1(0) = \tilde{r}_1 > 0$, and therefore

$$\left.\frac{d\ln r_2(\tau)}{p\ln r_1(\tau)}\right|_{\tau=0} = \left.\frac{d\ln\Phi_2(r_1)}{d\ln r_1}\right|_{r_1=\tilde{r}_1} = 0,$$

i.e. the curve

$$\ln r_2 = \ln\Phi_2(e^{\ln r_1}) = \chi(\ln r_1)$$

(decreasing, convex) would have a point of inflection at $r_1 = \tilde{r}_1 > 0$. Consequently, the curve $r_2 = \Phi_2(r_1)$ would not have the property of "logarithmic convexity" in the interval $0 < r_1 < R_1$. From the same formula we have $\Phi_2[\varphi(1)] = 0$, i.e. $r_2(1) = 0$ and $r_1(1) = R_1$. This is proved in exactly the same way; one need only consider $\ln r_1$ as a function of $\ln r_2$, since the inverse function

$$\ln r_1 = \chi_1(\ln r_2)$$

is monotonically decreasing and has the property of "logarithmic convexity." As for the condition

$$r_1'(\tau) \leqslant \frac{r_1(\tau)}{\tau},$$

it characterizes the convexity of the curve $r_2 = \Phi_2(r_1)$, and in this case the integral representations (3), (4) hold. If the curve $r_2 = \Phi_2(r_1)$ is not convex everywhere, i.e. the function

$$r_1'(\tau) - \frac{r_1(\tau)}{\tau}$$

on the interval $0 < \tau < 1$ either changes sign or is greater than zero, then it turns out that the domain bounded by the hypersurface $|z| = \Phi_2(|w|)$ can be mapped, by means of the pair of functions $w = W^n$, $z = Z^n$, where n is a positive integer, onto a domain whose boundary $|Z| = \Phi_3(|W|)$ has the property that the curve $r_3 = \Phi_3(r_1)$ is convex. Therefore, for the function $F(W^n, Z^n)$ the integral representations (3), (4) hold. Returning to the old variables, we obtain the integral representations (3′), (4′) for the original function $F(w, z)$ in the case when the curve $r_2 = \Phi_2(r_1)$ is not convex.

Corollary. *For any bounded, complete, bicircular domain whose boundary is twice continuously differentiable and analytically convex from the outside* $(L(\Phi_1) > 0)$*, the integral representations (3) and* (4′) *hold.*

For the validity of this assertion we need only include the case when the boundary $|z| = \psi(|w|)$ of the domain $\overline{D}$ has the property that the curve $r_2 = \psi(r_1)$ is convex or a straight line. But in this case

$$r_2'(\tau) \leqslant \frac{r_1(\tau)}{\tau},$$

and this is equivalent to the condition

$$\frac{d \ln r_1(\tau)}{d \ln \tau} \leqslant 1.$$

Hence $\mu \leqslant 1$, and therefore $n = 1$, i.e. the integral representations (3′), (4′) take, respectively, the form (3), (4), which proves the validity of our assertion.

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Note: Figure translations are in progress. See original paper for figures.