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Abstract

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MATHEMATICS

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ON THE UNIQUENESS OF THE GENERALIZED SOLUTION OF THE CAUCHY PROBLEM FOR SYSTEMS OF QUASILINEAR EQUATIONS OF HYPERBOLIC TYPE

(Presented by Academician M. V. Keldysh, 30 V 1958)

A **generalized solution of the Cauchy problem** for a conservative ([1]) system of quasilinear equations

$$\frac{\partial u_i}{\partial t} + \frac{\partial \varphi_i(u,t,x)}{\partial x} = 0 \quad (i = 1, 2, \dots, n); \qquad u = \{u_1, u_2, \dots, u_n\}, \tag{1}$$

is a system of functions $u_i(t,x)$ that take the prescribed values on the initial line $t = 0$ and satisfy the integral relations

$$\oint_C u_i(t,x)\,dx - \varphi_i(u(t,x),t,x)\,dt = 0 \quad (i = 1, 2, \dots, n), \tag{2}$$

where C is an arbitrary piecewise-smooth closed contour lying entirely in the half-plane $t \geq 0$.

The question of uniqueness of the generalized solution of the Cauchy problem for a single quasilinear equation has now been considered rather fully ([2,3]); for systems of quasilinear equations this question has been considered only for some special cases ([4,5]), and the methods used in doing so are not sufficiently general.

In considering the question of uniqueness of the generalized solution, we use the possibility of reducing the problem of finding a generalized (discontinuous) solution of the Cauchy problem for system (1) to the Cauchy problem for a system of nonlinear equations

$$\frac{\partial \Phi_i}{\partial t} + \varphi_i\left(\frac{\partial \Phi}{\partial x}, t, x\right) = 0 \quad (i = 1, 2, \dots, n); \tag{3}$$

$$\Phi_i(t,x) = \int_{(0,0)}^{(t,x)} u_i\,dx - \varphi_i(u,t,x)\,dt \quad (i = 1,2,\dots,n) \tag{4}$$

in the class of continuous solutions, as indicated by us in ([6]).

Here we restrict ourselves to proving the uniqueness theorem for the generalized solution of the Cauchy problem for a system of two quasilinear equations; however, the method we use is general and does not depend on the number of equations. In what follows we restrict ourselves to the class of piecewise-continuous and piecewise-smooth generalized solutions of the Cauchy problem for system (1), i.e., we exclude from consideration solutions with unbounded derivatives (centered rarefaction waves).

Suppose that the system of equations (1) is hyperbolic in the domain of variation of the variables u, t, x under consideration, i.e. the roots of the equation

$$\left| \frac{\partial \varphi_i}{\partial u_j} - \xi \delta_{ij} \right| = 0 \tag{5}$$

$$\xi_1(u,t,x), \dots, \xi_n(u,t,x)$$

are real and distinct in the domain under consideration. Let

$$\xi_1(u,t,x) < \xi_2(u,t,x) < \cdots < \xi_n(u,t,x). \tag{6}$$

Define, for the system of equations (1), the eigenvectors

$$c_k(u,t,x) = \{c_{k1}(u,t,x);\ c_{k2}(u,t,x); \dots;\ c_{kn}(u,t,x)\} \quad (k = 1,2,\dots,n)$$

by means of the relations:

$$\sum_{j=1}^{n} \frac{\partial \varphi_j}{\partial u_i} c_{kj} = \xi_k c_i \quad (i = 1,2,\dots,n); \qquad \sum_{j=1}^{n} c_{kj}^2 = 1 \quad (k = 1,2,\dots,n). \tag{7}$$

Equations (7), in view of (5), determine the eigenvectors up to sign, while conditions (6) ensure the linear independence of the vectors $c_1, c_2, \dots, c_n$.

Definition 1. We shall call the system of quasilinear equations (1), for $n = 2$, a **system of type (A)** if: 1) for any pair of points $(\bar{u}, t, x)$ and $(\bar{\bar{u}}, t, x)$ in the domain under consideration there is at least one point $(\tilde{u}, t, x)$ in this domain such that the equations

$$\varphi_i(\bar{u}, t, x) - \varphi_i(\bar{\bar{u}}, t, x) = \sum_{j=1}^{2} \frac{\partial \varphi_i}{\partial u_j}(\tilde{u}, t, x)(\bar{u}_j - \bar{\bar{u}}_j) \quad (i = 1, 2) \tag{8}$$

and the inequalities

$$[\xi_k(\bar{u}, t, x) - \xi_k(\tilde{u}, t, x)][\xi_k(\bar{\bar{u}}, t, x) - \xi_k(\tilde{u}, t, x)] \leq 0 \quad (k = 1, 2), \tag{9}$$

hold, and the right-hand sides of the equalities (8) are continuous and continuously differentiable functions of the variables $\bar{u}_1, \bar{u}_2, \bar{\bar{u}}_1, \bar{\bar{u}}_2, t, x$; 2) the eigenvectors $c_k(u, t, x)$ of the system of equations (1) satisfy, in the domain under consideration, the inequalities

$$|\Delta_{12}(\bar{u}, \bar{\bar{u}}, t, x)| > |\Delta_{11}(\bar{u}, \bar{\bar{u}}, t, x)|;$$

$$|\Delta_{12}(\bar{u}, \bar{\bar{u}}, t, x)| > |\Delta_{22}(\bar{u}, \bar{\bar{u}}, t, x)|, \tag{10}$$

where

$$\Delta_{ij}(\bar{u}, \bar{\bar{u}}, t, x) = \begin{vmatrix} c_{i1}(\bar{u}, t, x) & c_{i2}(\bar{u}, t, x) \\ c_{j1}(\bar{\bar{u}}, t, x) & c_{j2}(\bar{\bar{u}}, t, x) \end{vmatrix}. \tag{11}$$

Definition 2. We shall say that a piecewise-continuous and piecewise-smooth generalized solution of the system of quasilinear equations (1) **satisfies conditions (B)** if on each line of discontinuity of the solution $x = x(t)$ the inequalities of one of the following two groups are fulfilled:

$$\xi_1(u(t, x, (t) - 0), t, x(t)) < D < \xi_2(u(t, x(t) - 0), t, x(t)),$$

$$\xi_2(u(t, x(t) + 0), t, x(t)) < D \tag{12}$$

or else

$$D < \xi_1(u(t, x(t) - 0), t, x(t)),$$

$$\xi_1(u(t, x(t) + 0), t, x(t)) < D < \xi_2(u(t, x(t) + 0), t, x(t)), \tag{13}$$

where $D = x'(t)$ (*).

Theorem. The generalized solution of the Cauchy problem for the system of quasilinear equations (1) of type (A), satisfying conditions (B), is unique.

Here we shall say that **two generalized solutions coincide** if they coincide at points of continuity.

The restrictions imposed on systems of type (A) are overstrong and can be somewhat weakened. We do not touch here on the question of the independence of the various requirements imposed on systems of type (A). Let us note in this connection that the system of two equations of hydrodynamics

$$\frac{\partial u_1}{\partial t} + \frac{\partial \varphi_1(u_2, t, x)}{\partial x} = 0; \qquad \frac{\partial u_2}{\partial t} - \frac{\partial u_2}{\partial t} = 0; \qquad \frac{\partial^2 \varphi_1(u_2, t, x)}{\partial u_2^2} > 0, \quad (14)$$

considered in the domain $0 < \varepsilon \le u_2 \le M$, is a system of type (A).

Another simplest example of a system of type (A) is a decoupled system of two quasilinear equations

$$\frac{\partial u_1}{\partial t} + \frac{\partial \varphi_1(u_1, t, x)}{\partial x} = 0; \qquad \frac{\partial u_2}{\partial t} + \frac{\partial \varphi_2(u_2, t, x)}{\partial x} = 0,$$

if $\partial^2\varphi_1/\partial u_1^2$ and $\partial^2\varphi_2/\partial u_2^2$ do not change sign.

Proof. Suppose the contrary. Let $\bar{u}(t, x)$ and $\bar{\bar{u}}(t, x)$ be two distinct generalized solutions of the system of equations (1), satisfying conditions (B) and taking the same values for $t = 0$. They correspond to the potentials $\bar{\Phi}(t, x) = \{\bar{\Phi}_1(t, x); \bar{\Phi}_2(t, x)\}$ and $\bar{\bar{\Phi}}(t, x) = \{\bar{\bar{\Phi}}_1(t, x); \bar{\bar{\Phi}}_2(t, x)\}$, taking the same values for $t = 0$, satisfying the system of nonlinear equations (3), and being continuous functions of the variables t, x.

The difference of the potentials $v(t, x) = \{v_1(t, x); v_2(t, x)\} = \{\bar{\Phi}_1 - \bar{\bar{\Phi}}_1; \bar{\Phi}_2 - \bar{\bar{\Phi}}_2\}$ satisfies, for systems of type (A), the equations

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^{2} \frac{\partial \varphi_i}{\partial u_j}(\tilde{u}(t, x), t, x)\frac{\partial v_j}{\partial x} = 0 \qquad (i = 1, 2) \quad (15)$$

at all points where $\bar{u}(t, x)$ and $\bar{\bar{u}}(t, x)$ are continuous.

The system of equations (15) is hyperbolic; the characteristic directions are $\zeta_k(t, x) = \xi_k(\tilde{u}(t, x), t, x)$. The coefficients of the system of equations (15) have discontinuities together with $\bar{u}(t, x)$ and $\bar{\bar{u}}(t, x)$.

The linear system of equations (15) can be reduced to characteristic form. Introducing the Riemann invariants

$$r_k(t,x) = \sum_{i=1}^{2} c_{ki}(\tilde{u}(t,x),t,x)v_i(t,x) \qquad (k=1,2), \tag{16}$$

we obtain

$$\frac{\partial r_k}{\partial t} + \zeta_k(t,x)\frac{\partial r_k}{\partial x} = \sum_{i=1}^{2} B_{ik}(t,x)v_i(t,x) \qquad (k=1,2), \tag{17}$$

where $B_{ik}(t,x)$ are certain bounded functions.

Let us prove that in the domain G_{t_0}, defined by the inequalities

$$0 \le t \le t_0; \qquad x+a \ge M_1 t; \qquad x-a \le M_2 t; \tag{18}$$

$$M_1 = \sup_{G_{t_0}} \max\{\xi_2(\bar{u},t,x), \xi_2(\overline{\overline{u}},t,x)\};$$

$$M_2 = \inf_{G_{t_0}} \min\{\xi_1(\bar{u},t,x), \xi_1(\overline{\overline{u}},t,x)\},$$

$v_i(t,x) \equiv 0$. We assume that all discontinuity lines of the solutions $\bar{u}(t,x)$ and $\overline{\overline{u}}(t,x)$ in G_{t_0} issue from the point A with coordinates $(0,0)$.

Lemma 1. *A generalized solution $u(t,x)$, satisfying conditions (B), cannot have more than two discontinuity lines issuing from the point A. These two lines do not intersect in G_{t_0}. The right one satisfies inequalities (12), and the left one inequalities (13).*

Thus, the coefficients of the system of equations (17) have discontinuities in G_t on no more than four lines. Let these be the lines AL, AM, AN, AO. The domain G_t is divided by them into five zones (see Fig. 1). The solid lines depict the discontinuity lines of the solution $\bar{u}(t,x)$, and the dashed lines the discontinuity lines of $\overline{\overline{u}}(t,x)$. In proving uniqueness of the solution of the system of equations (15) we shall use Haar' s method ([8]).

It can be shown that all points of zones I and II lie in the domains of influence of the initial data of the intervals (A,a) and $(-a,A)$, respectively. Therefore, in zo-

in regions I and II of the domain G_{t_0}, $v_i(t,x) \equiv 0$ $(i=1,2)$; consequently, on the lines AL and AO, $r_k(t,x) = 0$ $(k=1,2)$. Further, one may assert that:

$$\text{in zone } III, \quad |r_1^{(3)}(t,x)| \le 2BVt_0, \tag{19}$$

Fig. 1

Figure 1: Fig. 1

$$\text{in zone } IV, \quad |r_2^{(4)}(t,x)| \le 2BVt_0, \tag{20}$$

where

$$V = \sup_{G_{t_0}} \max_i |v_i(t,x)|; \qquad B \ge |B_{ik}(t,x)| \quad (i,k = 1,2).$$

On the lines AM and AN the functions $r_k(t,x)$ are discontinuous; however, the functions $v_i(t,x)$ are continuous. From inequalities (19), (20) there follows the inequality

$$\left|\Delta_{12}(\overset{(5)}{u},\overset{(4)}{u},P)\Delta_{12}(\overset{(3)}{u},\overset{(5)}{u},S)r_2^{(5)}(R) - \Delta_{22}(\overset{(5)}{u},\overset{(4)}{u},P)\Delta_{11}(\overset{(3)}{u},\overset{(5)}{u},S)r_2^{(5)}(S)\right| \le DVt_0. \tag{21}$$

Here the superscript denotes the number of the zone to which the given discontinuous quantity belongs; P is some point of the line AN; the points R and S lie on the line AM; D is some constant.

Lemma 2. If a continuous function $r(t)$ satisfies, on the interval $0 \le t \le t_0$, the functional inequality

$$|\alpha(t)r(t) + \beta(t)r(\varphi(t))| \le \varepsilon; \qquad 0 < \varphi(t) < t; \qquad |\alpha(t)| \ge |\beta(t)| + \delta, \tag{21}$$

then

$$|r(t)| \le \varepsilon\alpha/\delta^2, \qquad \alpha \ge |\alpha(t)|.$$

Fig. 1

For the applicability of Lemma 2 to the estimate of $r_2^{(5)}(t,x)$, it is sufficient to show that

$$\left|\Delta_{12}(\overset{(5)}{u},\overset{(4)}{u},P)\Delta_{12}(\overset{(3)}{u},\overset{(5)}{u},S)\right| \ge \left|\Delta_{22}(\overset{(5)}{u},\overset{(4)}{u},P)\Delta_{11}(\overset{(3)}{u},\overset{(5)}{u},S)\right| + \delta, \qquad \delta > 0. \tag{22}$$

But the quantities entering inequality (23) are continuous functions of the points P and S. Since inequality (22) is satisfied when the points P and S coincide with the point A, in view of the requirements (10), it follows that, if t_0 is sufficiently small, inequality (22) will also be satisfied. Thus, for the quantity $r_2^{(5)}(t,x)$ we obtain the estimate

$$|r_2^{(5)}(t,x)| < \widetilde{D}Vt_0.$$

After this it is easy to obtain an analogous estimate for the solutions also in zones III, IV. We obtain that

$$|r_k(t,x)| < \widetilde{D}Vt_0 \qquad (k = 1, 2), \tag{23}$$

and the inequalities (23) hold throughout the domain G_{t_0}. Hence it follows that

$$|v_i(t,x)| < CVt_0 \quad (i = 1, 2),$$

where C is some constant. If t_0 is so small that $Ct_0 < 1$, then $v_i(t,x) \equiv 0$, and the theorem is proved.

We note that nothing in the proof of the theorem changes if one assumes that the lines of discontinuity of the solutions $\bar{u}(t,x)$ and $\bar{\bar{u}}(t,x)$ have a different mutual arrangement.

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