Reports of the Academy of Sciences of the USSR

K. F. VOITSEKHOVSKAYA

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1958. Volume 123, No. 4

THEORY OF ELASTICITY

K. F. VOITSEKHOVSKAYA

STABILITY OF CYLINDRICAL SHELLS FROM THE POINT OF VIEW OF THE MATHEMATICAL THEORY OF ELASTICITY

(Presented by Academician L. I. Sedov, 1 VII 1958)

The general problem of the stability of elastic bodies from the point of view of the equations of the nonlinear theory of elasticity was posed by V. V. Novozhilov ([1]). However, the solution of concrete problems has encountered difficulties that have not yet been overcome. Neglecting the components of rotation in the equilibrium equations, but retaining them in the boundary conditions of the problem, A. Yu. Ishlinskii ([2]) gave a solution of the problem of the stability of compression of an infinitely long strip under conditions of plane deformation. The value of the critical force found by him almost exactly coincides with that obtained from calculations by Euler' s formula for longitudinal bending.

In the present paper, the problem of the stability of compression of a cylindrical shell is considered on the basis of the equations of the mathematical theory of elasticity, likewise without taking account of the components of rotation in them, but with allowance for the deformation of the boundary surface of the body.

Let a hollow cylinder, whose inner and outer radii are respectively R_1 and R_2 (Fig. 1), be compressed in the axial direction by a force p, referred to a unit area. The lateral surface of the cylinder is free of forces. We shall find that value of p for which, in addition to the principal equilibrium position determined by the stresses

$$\sigma_r^0 = \sigma_\theta^0 = \tau_{zr}^0 = \tau_{\theta z}^0 = \tau_{r\theta}^0 = 0, \qquad \sigma_z^0 = -p \tag{1}$$

and displacements

$$u_r^0 = \frac{\nu p}{E}\, r, \qquad u_\theta^0 = 0, \qquad u_z^0 = -\frac{p}{E}\, z \tag{2}$$

Fig. 1

Figure 1: Fig. 1

Fig. 1

(where E is Young' s modulus, ν is Poisson' s ratio), another equilibrium state is possible, infinitely close to the principal one, in which the lateral surface of the cylinder is also free of forces, but may already be noncylindrical.

Let u_r, u_θ, u_z be additional infinitesimal displacements of the points of the cylinder from their positions in the initial deformed state; $\sigma_r, \sigma_\theta, \sigma_r, \tau_{r\theta}, \tau_{zr}, \tau_{\theta z}$ are the stresses corresponding to these displacements.

Consider the case of an axisymmetric form of loss of stability. Then the stress function U does not depend on θ, and the problem of stability of compression

of a hollow cylinder is reduced to the solution of the biharmonic equation

$$\nabla^4 U(r, z) = 0 \tag{3}$$

under the boundary conditions

$$(\sigma_r^0 + \sigma_r)\cos r\nu + (\tau_{rz}^0 + \tau_{rz})\cos z\nu = 0,$$

$$(\tau_{rz}^0 + \tau_{rz})\cos r\nu + (\sigma_z^0 + \sigma_z)\cos z\nu = 0$$

for

$$r = R_j + u_r(R_j, z) \qquad (j = 1, 2), \tag{4}$$

where $j = 1$ corresponds to the inner, and $j = 2$ to the outer lateral deformed surface of the cylinder; ν is the normal to the deformed surface; the stresses $\sigma_r, \sigma_\theta, \sigma_z$ and τ_{rz} are expressed in terms of the stress function by the known formulas ([3]).

Using the solution (1)–(2), the relation $\cos r\nu : \cos z\nu = 1 : (-\partial u_r/\partial z)$, and the expansions of the functions σ_r and τ_{rz} in Taylor series, it is easy to see that, to quantities of the second order of smallness, the boundary conditions (4) may be taken not on the deformed surface $r = R_j + u_r(R_j, z)$ $(j = 1, 2)$, but on the undeformed surface $r = R_j$ $(j = 1, 2)$; consequently the boundary conditions (4) take the form

$$\sigma_r = 0, \qquad \tau_{rz} = -p\,\frac{\partial u_r}{\partial z} \quad \text{for } r = R_j \quad (j = 1, 2). \tag{5}$$

The biharmonic equation (3) is satisfied by the function

$$U(r,z) = \left[C_1 J_0(iar) + C_2 r \frac{d}{dr} J_0(iar) + C_3 N_0(iar) + C_4 r \frac{d}{dr} N_0(iar)\right] \cos az, \tag{6}$$

where $J_0(iar)$, $N_0(iar)$ are respectively the Bessel and Neumann functions of zero order with imaginary argument iar; C_1, C_2, C_3 and C_4 are integration constants; the parameter a determines the wavelength, namely

$$a = \frac{m\pi}{l}, \qquad m = 1, 2, \ldots, \tag{7}$$

where l/m is the half-wavelength.

Using Hooke' s law and the boundary conditions (5), we obtain four linear algebraic equations for the unknowns C_1, C_2, C_3 and C_4:

$$\left[\frac{1}{t_j} J_1(t_j) - J_0(t_j)\right] C_1 + \left[t_j J_1(t_j) + (2\nu - 1) J_0(t_j)\right] C_2 +$$

$$+ \left[\frac{1}{t_j} N_1(t_j) - N_0(t_j)\right] C_3 + \left[t_j N_1(t_j) + (2\nu - 1) N_0(t_j)\right] C_4 = 0 \tag{8}$$

$$(j = 1, 2),$$

$$\left(p + \frac{E}{1+\nu}\right) J_1(t_j) C_1 + \left[\left(p + \frac{E}{1+\nu}\right) t_j J_0(t_j) + 2\frac{1-\nu}{1+\nu} E J_1(t_j)\right] C_2 +$$

$$+ \left(p + \frac{E}{1+\nu}\right) N_1(t_j) C_3 + \left[\left(p + \frac{E}{\nu+1}\right) t_j N_0(t_j) + 2\frac{1-\nu}{1+\nu} E N_1(t_j)\right] C_4 = 0 \quad (j = 1, 2),$$

where, for convenience, we have denoted

$$t_1 = iaR_1, \qquad t_2 = iaR_2. \tag{9}$$

A nontrivial solution of the system (8) exists under the condition that its determinant be equal to zero. This condition gives us an equation for determining the critical compressive force p_{cr}, at which loss of stability occurs, namely

$$A\left(p_{\text{cr}} + \frac{E}{1+\nu}\right)^2 + \xi B\left(p_{\text{cr}} + \frac{E}{1+\nu}\right) + \xi^2 C = 0, \tag{10}$$

where $\xi = \dfrac{2(1-\nu)}{1+\nu}E$, and the quantities A, B, and C are expressed in terms of Bessel and Neumann functions for $t = iaR_1$ and $t = iaR_2$.

Using, for large values of the argument $t = ix$, the asymptotic formulas (5) for cylindrical functions, one may represent A, B, and C in the form

$$\begin{aligned}
A &= \eta^2 \left\{ 4(s^2-1) + \frac{4}{3}s^2\eta^2 + \frac{8}{45}s^2\eta^4 + \frac{4}{315}s^2\eta^6 + \cdots \right. \\
&\quad \left. \cdots + \frac{\eta^2}{b^2}\left[\frac{1}{3}(s^2-1) + \frac{2}{45}(2s^2+2s-1)\eta^2 + \cdots\right] + o\left(\frac{\eta^4}{b^4}\right)\right\}, \\
B &= \eta^2 \left\{ -8s\left(1 + \frac{1}{3}\eta^2 + \frac{2}{45}\eta^4 + \frac{1}{315}\eta^6 + \cdots\right) + \right. \\
&\quad + \frac{1}{b^2}\left[8(1-s) + \frac{2}{3}(2-7s)\eta^2 + \frac{4(1-10s)}{45}\eta^4 + \cdots\right] + \\
&\quad \left. + \frac{\eta^2}{b^4}\left[2(1-s) + \cdots\right] + o\left(\frac{\eta^4}{b^6}\right)\right\}, \\
C &= \eta^2 \left\{ 4\left(1 + \frac{1}{3}\eta^2 + \frac{2}{45}\eta^4 + \frac{1}{315}\eta^6 + \cdots\right) + \frac{1}{b^2}\left(4 + 3\eta^2 + \frac{28}{45}\eta^4 + \cdots\right) + \right. \\
&\quad \left. + \frac{\eta^2}{b^4}(1+\cdots) + o\left(\frac{\eta^4}{b^6}\right)\right\},
\end{aligned} \tag{11}$$

where

$$\eta = \frac{m\pi}{l}(R_2 - R_1), \qquad b = \frac{m\pi}{l}\sqrt{R_1R_2}, \qquad s = 1 - 2\nu. \tag{12}$$

Since $A < 0$, $B < 0$, $C > 0$, one value of p_{cr} will be negative. The second value of p_{cr}, after using the relations (11) in equation (10), takes the form

$$\begin{aligned}
p_{\text{cr}} = E\Bigg\{ & \frac{\eta^2}{12(1-\nu^2)}\left[1 + \frac{7\nu-2}{60(1-\nu)}\eta^2 + \frac{61\nu^2-26\nu-1}{5040(1-\nu)^2}\eta^4 + \cdots\right] + \\
& + \frac{1}{b^2}\left[1 + \frac{1+\nu-\nu^2}{6(1-\nu^2)}\eta^2 + \frac{-5+25\nu-16\nu^2+6\nu^3}{720(1-\nu^2)(1-\nu)}\eta^4 + \cdots\right] + \\
& + \frac{4\nu(1-\nu)}{b^4}\left[1 + \frac{-1+7\nu+4\nu^2-6\nu^3}{48\nu(1-\nu^2)}\eta^2 + \cdots\right] - \frac{2\nu^3(1-\nu)}{b^6}[1+\cdots] + o\left(\frac{1}{b^8}\right)\Bigg\}.
\end{aligned} \tag{13}$$

Since the displacement u_r corresponding to the solution (3) vanishes for $z = 0$ and $z = l$, the formula obtained (13) can be compared with the formula of the theory of thin shells ([4]) for a cylindrical shell with hinged fastening at the edges

$$\sigma_{\text{cr}} = E\left[\frac{m^2\pi^2}{l^2}\frac{(R_2 - R_1)^2}{12(1-\nu^2)} + \frac{4l^2}{m^2\pi^2(R_1+R_2)^2}\right], \tag{14}$$

or, in the notation (12),

$$\sigma_{\text{cr}} = E\left[\frac{\eta^2}{12(1-\nu^2)} + \frac{4}{\eta^2+4b^2}\right]. \tag{15}$$

Assuming that many waves are formed along the length of the shell, and that σ_{cr} depends continuously on m, the minimum value of the critical load under the condition ([4])

$$\frac{\eta^2}{12(1-\nu^2)} = \frac{4}{\eta^2+4b^2} \tag{16}$$

is obtained in the form

$$\sigma_{\text{cr}} = \frac{E\eta^2}{6(1-\nu^2)}. \tag{17}$$

Using relation (16) in formula (13), we find

$$p_{\text{cr}} = \frac{E\eta^2}{6(1-\nu^2)}\left\{1 + \frac{8+35\nu-23\nu^2}{120(1-\nu^2)}\eta^2 + \frac{17+196\nu-69\nu^2-196\nu^3+192\nu^4}{5040(1-\nu^2)^2}\eta^4 + ...\right\}. \tag{18}$$

Thus, formula (17), derived by the methods of the theory of thin shells, is the limiting case as $\eta \to 0$ of formula (18), derived on the basis of the general equations of the mathematical theory of elasticity.

For $E = 2.08 \cdot 10^6$ kg/cm^2, $\nu = 0.3$, $\eta = 0.05$, we obtain

$$p_{\text{cr}} = 952.73928 \text{ kg/cm}^2; \qquad \sigma_{\text{cr}} = 952.38095 \text{ kg/cm}^2; \qquad p_{\text{cr}}/\sigma_{\text{cr}} = 1.000376.$$

The difference in the values of the critical stresses obtained from formulas (17) and (18) is on the order of hundredths of a percent.

In conclusion, I consider it my pleasant duty to express my deep gratitude to Academician of the Academy of Sciences of the Ukrainian SSR A. Yu. Ishlinskii for posing the problem and for his guidance.

Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

Received
20 V 1958

CITED LITERATURE

1. V. V. Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, Leningrad—Moscow, 1948, p. 147.

2. A. Yu. Ishlinskii, *Ukrainian Mathematical Journal*, 6, 140 (1954).

3. A. Love, *A Treatise on the Mathematical Theory of Elasticity*, Moscow—Leningrad, 1935, p. 288.

4. S. P. Timoshenko, *Stability of Elastic Systems*, Moscow, 1955, p. 445.

5. G. N. Watson, *Theory of Bessel Functions*, Foreign Literature Publishing House, 1949, p. 226.

Note: Figure translations are in progress. See original paper for figures.