INVESTIGATION OF A PROBLEM WITH AN OBLIQUE DERIVATIVE BY MEANS OF A SYSTEM OF FREDHOLM EQUATIONS

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Abstract

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INVESTIGATION OF A PROBLEM WITH AN OBLIQUE DERIVATIVE BY MEANS OF A SYSTEM OF FREDHOLM EQUATIONS

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1. In this article, a system of Fredholm integral equations will be brought to bear on the investigation of a problem with an oblique derivative. Restricting ourselves to the simply connected case, we may, without loss of generality, assume that the domain D, in which the solution of the problem is sought, is the unit disk with boundary Γ, since by means of a conformal mapping the problem can always be reduced to this case. For greater generality we shall consider a one-parameter family of problems B_λ.

Problem B_λ. Find a function $f = u + iv$, continuous in $D + \Gamma$, which is a generalized solution of the equation

$$\frac{\partial f}{\partial \bar{z}} = \lambda B(z)\overline{f(z)}, \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{1}$$

and which has a derivative $\partial f/\partial z$ continuously extendable to Γ, with the boundary condition on Γ being satisfied:

$$\operatorname{Re}\left[a\frac{\partial f}{\partial z} + \lambda b f\right] = \gamma, \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right). \tag{2}$$

Here $B(z)$ is a function prescribed in D, possessing a derivative $B_z \in L_p(D)$, $p > 2$; the functions a, b prescribed on Γ are continuously differentiable, while γ is H-continuous; λ is a real parameter. In view of the assumption $a(t) \neq 0$, $t \in \Gamma$, we may suppose that $|a| = 1$.

As shown in (1), this problem is equivalent to the following problem:

Problem C_λ. Determine a generalized solution, continuously extendable to Γ, of the system

$$\begin{aligned} \frac{\partial F_1}{\partial \bar z} &= \lambda B \overline{F}_1, \\ \frac{\partial F_2}{\partial \bar z} &= \lambda B_z \overline{F}_1 + \lambda^2 |B|^2 \overline{F}_1, \\ \frac{\partial F_3}{\partial \bar z} &= \overline{F}_2, \end{aligned} \tag{3}$$

satisfying on Γ the boundary condition

$$\operatorname{Re}[g_1(t)F(t)] = h_1(t),$$

$$g_1 = \begin{pmatrix} \lambda b & a & 0 \\ 1 & 0 & -1 \\ i & 0 & i \end{pmatrix}, \qquad F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

It is easy to show that problem C_λ can be reduced to a certain form in which, at the cost of a slight complication of the right-hand side of system (3), the matrix g_1 does not contain the element λb, while the function $a(t)$ entering it has the form $t^{-\varkappa}$, where $\varkappa$ is the index of the problems under consideration, equal to the increment of the function $\frac{1}{2\pi}[\arg a(t)]$ during one circuit of t along Γ in the positive direction. Indeed, denoting by $\beta(z)$ an arbitrary extension inside D of the function b/a given on Γ, subject to the condition $\beta_z \in L_p(D)$, $p > 2$, and introducing a new unknown function $\widetilde{\varphi}_2$ by the formula $\widetilde{\varphi}_2 = F_2 + \lambda\beta F_1$, we obtain the first simplification in condition (4). Next denote by $\Omega(z)$ an analytic function in D whose real part has on Γ the values $\arg a(t) + \varkappa \arg t$; these values are single-valued by virtue of the choice of the number $\varkappa$, and $\Omega(z)$ is constructed from them by the known Schwarz formula. Making one more change of unknowns by the formulas $\varphi_1 = F_1$, $\varphi_2 = e^{-i\Omega}\widetilde{\varphi}_2$, $\varphi_3 = F_3$, we obtain, as simple computations show, instead of (3) the following system:

$$\frac{\partial \varphi}{\partial \bar z} = A_\lambda \varphi + B_\lambda \bar\varphi,$$

$$A_\lambda = \begin{pmatrix} 0 & 0 & 0 \\ (\lambda^2|B|^2 + \lambda\beta_{\bar z})e^{i\Omega} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_\lambda = \begin{pmatrix} \lambda B & 0 & 0 \\ (\lambda^2 \beta B + \lambda B_z)e^{i\Omega} & 0 & 0 \\ -\lambda\bar\beta & e^{i\Omega} & 0 \end{pmatrix}, \tag{5}$$

and the boundary condition (4) is replaced by the following one:

$$\operatorname{Re}[g(t)\varphi(t)] = h(t),$$

$$g(t) = \begin{pmatrix} 0 & t^{-\varkappa} & 0 \\ 1 & 0 & -1 \\ i & 0 & i \end{pmatrix}, \qquad h = \begin{pmatrix} \gamma e^{\omega_\varkappa} \\ 0 \\ 0 \end{pmatrix}, \tag{6}$$

where $\omega_\varkappa = \operatorname{Im}\Omega(z)$. Thus, the boundary condition (6) of problem C_λ does not contain the parameter λ.

2. Let us first assume that the index of the problems $\varkappa$ is nonnegative ($\varkappa \geq 0$), and consider the matrix $G(t)$, defined on Γ by the formula

$$G(t) = g^{-1}(t)\,\overline{g(t)} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & t^{2\varkappa} & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{7}$$

By virtue of the assumption $\varkappa \geq 0$, this matrix admits an extension inside the domain D as a continuous matrix analytic in z.

Let us now consider the operator

$$T_\lambda \varphi \equiv -\frac{1}{\pi} \iint_D \left[\frac{\omega(t)}{t - z} + G(z) \frac{\overline{z\omega(t)}}{1 - \bar{t}z} \right] d\sigma_t, \tag{8}$$

$$\omega(t) \equiv A_\lambda \varphi + B_\lambda \bar{\varphi}.$$

The most important property of this operator is that it maps any set of vector-functions $\varphi(z) \in L_p(D)$, $p > 2$, into a set of vector-functions satisfying the condition (on the contour Γ)

$$\operatorname{Re}[g(t) T\varphi(t)] = 0, \qquad t \in \Gamma, \tag{9}$$

which is easily verified on the basis of relation (7). Moreover, it is linear over the field of real numbers, and is completely continuous in the real Hilbert space of three-component vector-functions with scalar product

$$\left(\varphi^{(1)}, \varphi^{(2)}\right) = \operatorname{Re} \iint_D \sum_{i=1}^{3} \varphi_i^{(1)} \varphi_2^{(i)} \, d\sigma$$

and represents one of the primitives with respect to the operator $\partial/\partial z$:

$$\partial T\varphi / \partial z = \omega(z).$$

By virtue of the properties of the operator T_λ listed above, in order that the vector-function φ be a solution of system (5), it is necessary and sufficient that it satisfy the system of integral equations

$$\varphi(z) = T_\lambda \varphi(z) + \Phi(z), \tag{10}$$

where Φ is a vector of analytic functions continuous in $D + \Gamma$. By virtue of relation (9), in order that a solution of system (10) be a solution of problem C_λ, it is necessary and sufficient that Φ be a solution of problem (6) in the class of analytic functions. This problem is solved effectively, namely

$$\Phi_1 = \alpha + i\beta, \qquad \Phi_3 = \alpha - i\beta,$$

where α and β are arbitrary real constants, while the component Φ_2, satisfying the condition

$$\operatorname{Re} t^{-\varkappa} \Phi_2 = \gamma_1, \qquad \gamma_1 = \gamma e^{\omega_x},$$

is recovered by means of the Schwarz integral (by formula (3), § 8):

$$\Phi_2(z) = \frac{z^\varkappa}{2\pi i} \int_\Gamma \gamma_1(t) \frac{t+z}{t-z} \frac{dt}{t} + i\tilde{\alpha} z^\varkappa + \sum_{k=0}^{\varkappa-1} \{\alpha_k (z^k z^{2\varkappa-1}) + i\beta_k (z^k + z^{2\varkappa-k})\}; \tag{11}$$

here $\alpha, \alpha_k, \beta_k,\ k = 0, 1, \dots, \varkappa - 1$, are arbitrary constants, so that the vector Φ contains $2\varkappa + 3$ arbitrary real parameters.

Thus, problem C_λ is reduced to the solution of system (10) with the free term found. As examples of problems of type B show (^{2}, § 75, 3°), the operator T_λ may have a nonempty spectrum S^+. However, one can show that this spectrum is always discrete. This follows from the fact that the number $\lambda = 0$, as can be shown from the preceding relations, is not an eigenvalue of the operator T_λ, and then, by virtue of one theorem from (^{4}), its resolvent exists and is meromorphic in λ.

We shall call the set S^+ the spectrum of the problems B_λ and C_λ for $\varkappa \geq 0$. We can formulate the following assertion.

Theorem 1. *Suppose that the index of the problems B_λ and C_λ is nonnegative ($\varkappa \geq 0$). Then the spectrum S^+ of these problems is always discrete, and if λ is not an eigenvalue, then the homogeneous problems $\overset{0}{B}_\lambda$ and $\overset{0}{C}_\lambda$ have exactly $2\varkappa + 3$ linearly independent solutions over the field of real numbers, while the nonhomogeneous problems B_λ and C_λ are solvable for any right-hand side. If $\lambda \in S^+$, then the number l of linearly independent solutions of the homogeneous problems $\overset{0}{B}_\lambda$ and $\overset{0}{C}_\lambda$ is equal to $l_* + 2\varkappa + 3$, and the number l_* of linearly independent solutions of the homogeneous adjoint problem (^{1}) satisfies the inequalities $l_* \leq k$ and $l \geq k - 3$ for $k > 3$, where k is the multiplicity of the eigenvalue λ.*

The last assertion follows from Theorem 4 (^{1}) and some additional considerations.

3. Let us now consider the case of negative index:

$$\varkappa_1 = -\varkappa < 0 \qquad (\varkappa > 0).$$

The preceding constructions are not applicable in this case, since matrix (7) cannot be continued inside D regularly and analytically, so that an operator of type (8) does not exist. Nevertheless, the system of integral equations to the study of which problem C_λ is reduced can still be obtained (cf. (5)). Namely, let us construct the operator

$$K_\lambda \varphi = -\frac{1}{\pi} \iint_D \left[\frac{\omega(t)}{t - z} + \overline{G}(t) \frac{\overline{\omega(t)}}{\overline{t}(1 - \overline{t}z)} \right] d\sigma_t. \tag{12}$$

It possesses all the properties inherent in the operator (8), with the exception of property (9). We shall call its spectrum S^- the spectrum of the problems B_λ and C_λ for negative index $\varkappa_1 < 0$.

Theorem 2. Suppose that the index of the problems B_λ and C_λ is negative ($\varkappa_1 = -\varkappa,\ \varkappa > 0$). Every solution of the problem C_λ also satisfies the system of integral equations:

$$\varphi(z) = K_\lambda \varphi(z) + \Phi(z), \tag{13}$$

where the vector Φ, holomorphic and continuous in $D + \Gamma$, is determined by the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_\Gamma g^{-1}(t) \frac{h(t)\, dt}{t - z} - G(0)\overline{\varphi(0)} \tag{14}$$

(g and G have the same meanings (6) and (7) as above). The spectrum S^- of these problems is also always discrete, and if λ is not an eigenvalue, then the number l of solutions, linearly independent over the field of real numbers, of the homogeneous problems B_λ and C_λ^0 does not exceed two; moreover, depending on the function $B(z)$, the cases $l = 1$, $l = 2$ are possible.

If $\lambda \notin S^-$, then the system (13) is solvable for any free term $\Phi(z)$; however, its solution does not, in general, give a solution of the nonhomogeneous problem C_λ with right-hand side h entering into (14). In addition, when seeking solutions of the problem C_λ, it is natural to normalize them at the origin by the condition $\varphi_1(0) = \alpha + i\beta$, where α and β are certain real constants, since, as follows from Theorem 2, cases of nonunique solvability of C_λ are possible. The function γ_1 prescribed on Γ has the form of a sum $\gamma^{(1)} + \gamma^{(2)}$, where $\gamma^{(1)}$ belongs to the subspace $H_2^{(1)}(\Gamma)$ of $H_2(\Gamma)$ spanned by the basis $1, \cos\varphi, \sin\varphi, \dots, \cos\varkappa\varphi, \sin\varkappa\varphi$, while $\gamma^{(2)}$ belongs to its orthogonal complement. As follows from (14), the component $\gamma^{(1)}$ has no effect on the solution: it is determined exclusively by the component $\gamma^{(2)}$ and by the normalization at the origin.

Theorem 3. There exist three functions $\nu(t)$, $\mu^{(1)}(t)$, $\mu^{(2)}(t)$, belonging to the subspace $H_2^{(1)}(\Gamma)$ (where ν is completely determined by the component $\gamma^{(2)}$, while $\mu^{(1)}$, $\mu^{(2)}$ do not depend on γ_1), such that the nonhomogeneous problems C_λ and B_λ for $\lambda \notin S^-$ have a unique solution, normalized by the condition $\varphi_1(0) = \alpha + i\beta$, with the modified function

$$\gamma^* = \gamma^{(2)}(t) + \nu(t) + \alpha\mu^{(1)}(t) + \beta\mu^{(2)}(t)$$

in the corresponding right-hand sides of the problems B_λ and C_λ.

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Note: Figure translations are in progress. See original paper for figures.

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