# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON THE ASYMPTOTICS OF THE $\epsilon$-ENTROPY OF ANALYTIC FUNCTIONS

*(Presented by Academician A. N. Kolmogorov, January 27, 1958)*

1. Let $K$ be a continuum in the plane of the complex variable $z$, and let $G$ be a domain containing $K$. Denote by $A_G^K$ the normed space whose points are all functions $f(z)$, single-valued and analytic in the domain $G$, with the norm $\|f\|$ taken to be $\max_{z\in K}|f(z)|$. If, in addition, a constant $M > 0$ is fixed, then in the space $A_G^K$ one distinguishes the compact set $A_G^K(M)$, consisting of all functions $f(z)$ for which $\sup_{z\in G}|f(z)| \leq M$. As usual, $H_\varepsilon(F)$ denotes the $\varepsilon$-entropy of the compact set $F$. We are interested in the problem of the asymptotics, as $\varepsilon \to 0$, of the quantity $H_\varepsilon(A_G^K(M))$. The answer to this problem does not depend on which precise meaning for the symbol $H_\varepsilon$ we choose among those assigned to it by A. N. Kolmogorov (1), and also by A. G. Vitushkin (2). One may suppose, for example, that $H_\varepsilon = \log N_\varepsilon$, where $N_\varepsilon$ is the number of points of a minimal $\varepsilon$-net of the compact set*.

After the first result in this direction by A. N. Kolmogorov (1):

$$0 < \tau_1(K/G) < H_\varepsilon(A_G^K(M))\left(\log\frac{M}{\varepsilon}\right)^{-2} < \tau_2(K/G) < +\infty^{**}$$

there naturally arose the question: in what cases does the limit

$$\lim_{\varepsilon\to 0} H_\varepsilon(A_G^K(M))\left(\log\frac{M}{\varepsilon}\right)^{-2}$$

exist, and how is it to be found.

Some particular results in the problem under consideration were recently obtained by A. G. Vitushkin (2). These results of A. G. Vitushkin are completely contained in Theorems 1, 2, and 3 reported here.

2. In what follows, the continuum $K$ is assumed to be different from a single point and from the whole plane. $\{D_q\}$ is a sequence of domains adjacent to $K$. An arbitrary domain $G$ containing $K$ is called an "elementary neighborhood of $K$" if each of the domains $G_q = D_q \cap G$ is at most doubly

connected. The neighborhood is called nondegenerate if it differs from the whole plane with at most finitely many points removed. By $R_q$ we denote the modulus of the doubly connected domain $G_q$ $(1 < R_q \leq +\infty)$; if $G_q$ is simply connected, then we put $R_q = +\infty$.

**Theorem 1.** *Let $K$ be an arbitrary continuum and let $G$ be an arbitrary elementary nondegenerate neighborhood of it. Then the asymptotic formula holds*

* All logarithms are taken to base 2.
** The domain $G$ must not degenerate.

$$H_\varepsilon\left(A_G^K(M)\right) \simeq \tau(K/G)\left(\log\frac{M}{\varepsilon}\right)^2,$$

where

$$\tau(K/G) = \sum_q \frac{1}{\log R_q}.$$

The proof will be outlined in §§ 5–8.

Let us note here one more inequality. Let $F$ be a closed set of capacity $c$, and let $G$ be an arbitrary disk of radius $r$ containing $F$. Then

$$H_\varepsilon\left(A_G^F(M)\right) \geq \frac{1}{\log\frac{r}{c}}\left(\log\frac{M}{\varepsilon}\right)^2.$$

3. Consider the space of $n$ complex variables $z_1, z_2, \dots, z_n$, and in this space a domain $G$ and a continuum $K \subset G$. In this general formulation, the question of the asymptotics of $H_\varepsilon\left(A_G^K(M)\right)$ remains open. We can give here only a result pertaining to the case of polycylindrical domains and continua.

**Theorem 2.** Let in each of the planes $z_p$ $(p = 1, 2, \dots, n)$ an arbitrary continuum $K_p$ and an arbitrary elementary nondegenerate neighborhood $G_p$ of it be fixed. Put $K = K_1 \times K_2 \times \dots \times K_n$, $G = G_1 \times G_2 \times \dots \times G_n$. Then

$$H_\varepsilon\left(A_G^K(M)\right) \simeq \tau_n(K/G)\left(\log\frac{M}{\varepsilon}\right)^{n+1},$$

where

$$\tau_n(K/G) = \frac{2}{(n+1)!}\prod_{p=1}^{n}\tau(K_p/G_p).$$

This theorem is obtained by induction on $n$ with the help of limiting relations established in the proof of Theorem 1.

4. Consider the space $E^K_{\sigma,s}(c)$ of entire functions $f(z_1, z_2, \dots, z_n)$ of order $s = (s_1, s_2, \dots, s_n)$ and type $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$, more precisely, the class of all functions $f(z)$ for which the inequality

$$|f(z)| \leq c \exp[\sigma_1 |z_1|^{s_1} + \sigma_2 |z_2|^{s_2} + \cdots + \sigma_n |z_n|^{s_n}]$$

holds for every $z = (z_1, z_2, \dots, z_n)$; here $\|f\| = \max_{z \in K} |f(z)|$, $K$ is a bounded continuum. We assume that $\sigma_p > 0$, $s_p > 0$ $(p = 1, 2, \dots, n)$, since otherwise the matter reduces to a smaller number of variables. The following result is a simple generalization of the corresponding formula of A. G. Vitushkin, proved by him in the case when $K$ is a product of disks.

**Theorem 3.** Let $K$ be an arbitrary bounded continuum in $n$-dimensional complex space. Let $i_1, i_2, \dots, i_m$ be the indices of all coordinate planes $(z_p)$ onto each of which the projection of $K$ is not a point. Then

$$H_\varepsilon\left(E^K_{\sigma,s}(c)\right) \simeq \frac{2}{(m+1)!} s_{i_1} s_{i_2} \cdots s_{i_m} \left(\log \frac{c}{\varepsilon}\right)^{m+1} \left(\log \log \frac{c}{\varepsilon}\right)^{-m}.$$

5. The method of proof of Theorem 1 is contained in my note ($^3$). There it is shown how to construct a sequence of functions $e_{qn}(z)$, analytic in $G$, such that:

1) $f(z) = a_{00} + \sum_q \sum_{n=1}^{\infty} a_{qn} e_{qn}(z)$ for every $f(z)$ analytic in $G$.

2) The expansion 1) is unique; this point is very important for us.

3) The following properties of $e_{qn}(z)$ and the coefficients $a_{qn}$ hold:

1°. $\|e_{qn}\| < C(\delta)(1+\delta)^n$.

2°. $\sup\limits_{z \in G} |e_{qn}(z)| < C(\delta)(1+\delta)^n R_q^n$.

3°. $|a_{qn}| < \dfrac{M}{R_q^n} C(\delta)(1+\delta)^n$.

4°. $|a_{qm}| < C\|f\|^*$.

($\delta > 0$ is arbitrary, $q$ and $n$ arbitrary; $C$ and $C(\delta)$ depend only on $K$ and $G$.)

6. We shall use as our initial concept the "absolute" $\varepsilon$-entropy:

$$H_\varepsilon = \log N_\varepsilon,$$

where $N_\varepsilon$ is the least number of sets of diameter not greater than $2\varepsilon$ into which the compact set can be divided. Then, obviously,

$$N_\varepsilon^{(c)} \leq N_\varepsilon \leq N_\varepsilon^{(a)},$$

where $N_\varepsilon^{(c)}$ is the number of points in an arbitrary $2\varepsilon$-chain lying in the compact set (an $\alpha$-chain means any set whose points are pairwise separated by more than $\alpha$), and $N_\varepsilon^{(a)}$ is the number of points in an arbitrary $\varepsilon$-net for the compact set, with the elements of the net themselves belonging to an arbitrary metric enlargement of the compact set (in our case such an enlargement will be the space $A_G^K$).

7. Put

$$A_{qn}^{(c)} = \frac{1}{4} M \frac{1}{(q+1)^{2n^2} C(\delta)(1+\delta)^n R_q^n} \qquad (q \geq 0,\ n \geq 1).$$

In the disk $|a_{qn}^{(c)}| \leq A_{qn}^{(c)}$ take

$$N_{q,n,\varepsilon}^{(c)} > (A_{qn}^{(c)}/2\varepsilon C)^2$$

points, pairwise separated by more than $2\varepsilon C$. Let these points be

$$a_{qn,m_{qn}}^{(c)} \qquad (m_{qn} = 1, 2, \dots, N_{q,n,\varepsilon}^{(c)}).$$

The number $n_{q,\varepsilon}^{(c)}$ is defined as the last of the numbers $n$ for which

$$A_{qn}^{(c)} > 2\varepsilon C.$$

Consider all sums of the form

$$S^{(c)}(z) = \sum_q \sum_{n=1}^{n_{q,\varepsilon}^{(c)}} a_{qn,m_{qn}}^{(c)} e_{qn}(z).$$

It follows from 2° that

$$\sup |S^{(c)}(z)| \leq M,$$

and from 4° and 2), that

$$\|S_1^{(c)}(z) - S_2^{(c)}(z)\| > 2\varepsilon$$

for any two distinct sums of the form under consideration. In other words, the sums $S^{(c)}(z)$ form a $2\varepsilon$-chain in the compact set $A_G^K(M)$. Hence we conclude that

$$H_\varepsilon(A_G^K(M)) \geq \sum_q \sum_{n=1}^{n_{q\varepsilon}^{(c)}} \log N_{q,n,\varepsilon}^{(c)} \geq \cdots$$

$$\cdots \geq \sum_q \frac{1}{\log[R_q(1+\delta)]} \left(\log \frac{M}{\varepsilon}\right)^2 - C_1(\delta) \log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon}$$

for sufficiently small $\varepsilon$. Therefore

$$\lim_{\varepsilon\to 0} H_\varepsilon \left(\log \frac{M}{\varepsilon}\right)^{-2} \geq \sum_q \frac{1}{\log[R_q(1+\delta)]}.$$

*

In 4° it is assumed that the length of the boundary of the domain $D_q$ $(q = 0, 1, \ldots)$ is finite. The same is assumed in item 7; the general case can be obtained by a suitable limiting passage.

Finally, since the left-hand side does not depend on $\delta$, and since $\delta$ is arbitrary:

$$\underline{\lim}_{\varepsilon\to 0} H_\varepsilon(A_G^K(M)) \left(\log \frac{M}{\varepsilon}\right)^{-2} \geqslant \sum_q \frac{1}{\log R_q}.$$

8. As indicated in (3), for any function $f(z) \in A_G^K$ and for any $\delta > 0$ there exists a constant $A(\delta)$, depending only on $\sup_{z\in G} |f(z)|$, if the function is bounded in $G$, and such that

$$\left\| f(z) - a_{00} - \sum_q \sum_{n=1}^{n_q} a_{qn} e_{qn}(z) \right\| \leqslant A(\delta) \sum_q \left(\frac{1+\delta}{R_q}\right)^{n_q}.$$

Choose $n_{q,\varepsilon}^{(a)}$ so that $((1+\delta)/R_q)^{n_{q,\varepsilon}^{(a)}} < \varepsilon/4A(\delta)(q+1)^2$, but for $n_{q,\varepsilon}^{(a)} - 1$ this inequality is not yet satisfied. Put

$$A_{qn}^{(a)} = MC(\delta)(1+\delta)^n/R_q^n.$$

In the circle $|a_{qn}| \leqslant A_{qn}^{(a)}$ take

$$N_{q,n,\varepsilon}^{(a)} \leqslant (2A_{qn}^{(a)}/\varepsilon_{qn})^2$$

points forming an $\varepsilon_{qn}$-net for this circle, where

$$\varepsilon_{qn} = \varepsilon/12(q+1)^2(n+1)^2 C(\delta)(1+\delta)^n.$$

Let these points be

$$a_{qn,m_{qn}}^{(a)} \quad (m_{qn} = 1, 2, \ldots, N_{q,n,\varepsilon}^{(a)}).$$

Now, using 1° and 3°, we see that all possible sums

$$S^{(a)}(z) = a_{0,0,\varepsilon}^{(a)} + \sum_q \sum_{n=1}^{n_{q,\varepsilon}^{(a)}} a_{qn,m_{qn}}^{(a)} e_{qn}(z)$$

form an $\varepsilon$-net for the compact set $A_G^K(M)$. Hence

$$H_\varepsilon(A_G^K(M)) \leqslant \log N_{0,0,\varepsilon}^{(a)} + \sum_q \sum_{n=1}^{n_{q,\varepsilon}^{(a)}} \log N_{q,n,\varepsilon}^{(a)} \leqslant \cdots$$

$$\cdots \leqslant \sum_q \frac{1}{\log|R_q/(1+\delta)|} \left(\log \frac{M}{\varepsilon}\right)^2 + C_2(\delta) \log \frac{1}{\varepsilon} \cdot \log\log \frac{1}{\varepsilon}.$$

Therefore

$$\overline{\lim_{\varepsilon\to 0}}\, H_\varepsilon \left(\log \frac{M}{\varepsilon}\right)^{-2} \leqslant \sum_q \frac{1}{\log|R_q/(1+\delta)|},$$

and, finally, “putting $\delta = 0$,” we obtain

$$\overline{\lim_{\varepsilon\to 0}}\, H_\varepsilon(A_G^K(M)) \left(\log \frac{M}{\varepsilon}\right)^{-2} \leqslant \sum_q \frac{1}{\log R_q}.$$

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## CITED LITERATURE

[1] A. N. Kolmogorov, DAN, **108**, No. 3, 385 (1956).
[2] A. G. Vitushkin, DAN, **117**, No. 5 (1957).
[3] V. D. Erokhin, DAN, **120**, No. 4 (1958).

*Note: Figure translations are in progress. See original paper for figures.*

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