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Abstract

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ON THE SPECTRUM OF GROWING FUNCTIONS

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We consider the normed ring L_φ of functions* $f(x)$ with norm

$$\|f\| = \int_{-\infty}^{\infty} |f(x)|\varphi(x)\,dx,$$

where $\varphi(x)$ satisfies the conditions**: 1) $\varphi(x) \geqslant 1$;
2) $\varphi(x+y) \leqslant \varphi(x)\varphi(y)$; 3)

$$\int_{1}^{\infty} \frac{\psi(x)}{x}\,dx < \infty \qquad \left(\psi(x) = \sup_{|t|\geq x\geq 1} \frac{\ln\varphi(t)}{t}\right).$$

Denote by M_φ the space conjugate to L_φ; it consists of the functions $g(x)$ for which

$$\operatorname{vrai\,max} \frac{|g(x)|}{\varphi(x)} < \infty.$$

A point λ of the real axis will be called a point of the spectrum Λ_g^{I} of the function $g(x) \in M_\varphi$, if at this point the Fourier transforms of all functions $f(x) \in L_\varphi$ satisfying the equality

$$\int_{-\infty}^{\infty} f(x-t)g(x)\,dx = 0 \quad (-\infty < t < \infty) \tag{1}$$

vanish.

Following Beurling (^{2}), a point λ of the real axis will be called a point of the spectrum Λ_g^{II} of the function $g(x) \in M_\varphi$, if for all $\varepsilon > 0$ the inequality

$$\lim_{y\to+0} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |U_g(x,y)|\,dx > 0$$

holds, where

$$U_g(x, y) = \int_{-\infty}^{\infty} g(t) e^{-y|t|} e^{itx}\, dt$$

is the harmonic Fourier transform of the function $g(x)$ in the half-plane $y > 0$. The function $U_g(x, y)$ exists for $y > 0$, since from 2) and 3) it follows that $\psi(x) \to 0$ as $x \to \infty$. If λ does not belong to the spectrum Λ_g^{II}, then the functions

$$G^+(z) = \int_0^{\infty} g(t) e^{itz}\, dt, \qquad \operatorname{Im} z > 0; \tag{2}$$

$$G^-(z) = -\int_{-\infty}^{0} g(t) e^{itz}\, dt, \qquad \operatorname{Im} z < 0, \tag{2'}$$

are analytic continuations of one another through the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$. In this paper we investigate the question of the relation between the spectra Λ_g^I and Λ_g^{II}.

Theorem 1. Let $\varphi(x)$ satisfy conditions 1), 2), 3) and let $g(x) \in M_\varphi$; then $\Lambda_g^I = \Lambda_g^{\mathrm{II}}$.

Proof. First we prove that $\Lambda_g^{\mathrm{II}} \supset \Lambda_g^I$. Indeed, let λ not be a point of the spectrum Λ_g^{II}, i.e., let the functions $G^+(z)$ and $G^-(z)$ admit analytic continuation through some segment $[\lambda - \varepsilon, \lambda + \varepsilon]$. From 3) there follows the existence of a function $f_1(t) \in L_\varphi$ whose Fourier transform $F_1(x)$ is equal to zero outside the interval $(\lambda - \varepsilon, \lambda + \varepsilon)$ and $F_1(\lambda) \neq 0$. Parseval' s equality gives

$$\int_0^{\infty} f(x-t) g(x) e^{-yx}\, dx = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} F_1(x) G^+(x+iy) e^{itx}\, dx,$$

$$-\int_{-\infty}^{0} f(x-t) g(x) e^{yx}\, dx = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} F_1(x) G^-(x-iy) e^{itx}\, dx \qquad (f(x) = f_1(-x)).$$

As $y \to +0$ the limits of the right-hand sides are equal; consequently,

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* For a similar ring, see (^{1}).

** Condition 3) may be replaced by the following: 3)

$$\int_{-\infty}^{\infty} \frac{\ln \varphi^*(x)}{1+x^2}\, dx < \infty, \quad \text{where } \varphi^*(x) = \sup_{|t| \le x} \varphi(t).$$

$$\int_0^\infty f(x-t)g(x)\,dx = -\int_{-\infty}^0 f(x-t)g(x)\,dx,$$

whence the validity of equality (1) follows and, therefore, $\lambda \notin \Lambda_g^I$.

The proof that $\Lambda_g^{\text{II}} \subset \Lambda_g^I$ is based on two lemmas.

Lemma 1. If the function $\varphi(x)$ satisfies conditions 1)–3), then for the function

$$M(y) = \int_1^\infty \varphi(x) e^{-2xy}\,dx$$

the condition

$$\int_0^k \ln\ln M(y)\,dy < \infty, \qquad \text{where } k = \sup_{1\le x<\infty} \frac{\ln\varphi(x)}{x}.$$

is satisfied.

Proof. Without loss of generality, we shall assume the function $\psi(x)$ to be continuous. Denote by $\alpha(y)$ the function inverse to $\psi(x)$. It is clear that $\alpha(y)$ decreases monotonically, and $\alpha(y) \to \infty$ as $y \to 0$. We shall show that

$$\int_0^k \ln\alpha(y)\,dy < \infty. \tag{3}$$

Indeed, after integration by parts we shall have:

$$\int_\varepsilon^k \ln\alpha(y)\,dy \le \int_k^\varepsilon \frac{y}{\alpha(y)}\,d\alpha(y) + k\ln\alpha(k) \le \int_1^\infty \frac{\psi(x)}{x}\,dx + k\ln\alpha(k) < \infty$$

for any $\varepsilon > 0$. The function $M(y)$ can be estimated in the following way:

$$M(y) \le \int_1^{\alpha(y)} e^{x\psi(x)}e^{-2xy}\,dx + \int_{\alpha(y)}^\infty e^{x\psi(x)}e^{-2xy}\,dx \le$$

$$\le \int_1^{\alpha(y)} e^{kx}\,dx + \int_0^\infty e^{-xy}\,dx \le \frac{1}{k}e^{k\alpha(y)} + \frac{1}{y}.$$

The last inequality, together with (3), proves the lemma.

Lemma 2 (Levinson*). Let $M(y)$ be a positive monotonically decreasing function on the interval $(0,1)$, with $M(y) \to \infty$ as $y \to 0$. Let $f(z)$ be a function analytic in the rectangle $|x| \leq a,\ |y| \leq b$, and let, in this rectangle, $|f(z)| < M(|y|)$.

If

$$\int_0^1 \ln \ln M(y)\, dy < \infty,$$

then there exists a constant M, depending only on $M(y)$ and $\delta > 0$, such that in the rectangle $|x| \leq a - \delta,\ |y| \leq b$

$$|f(x+iy)| < M.$$

We shall now prove that $\Lambda_g^I \supset \Lambda_g^{\text{II}}$. Suppose that relation (1) holds and that the Fourier transform of the function $f(x)$ is different from zero on the interval (a, b). It is necessary to show that the functions $G^+(z)$ and $G^-(z)$, defined by the equalities (2) and (2′), are analytic continuations of one another through the interval (a, b). To this end we construct a family of functions $\mu_\delta(x) \in L_\varphi$ satisfying the conditions: 1°. $|\mu_\delta(x)| \leq 1$. 2°. $\lim_{\delta \to 0} \mu_\delta(x) = 1$. 3°. The Fourier transform of the function $\mu_\delta(x)$ vanishes outside the interval $(-\delta, \delta)$.

If $\lambda \in (a, b)$ and δ is so small that the interval $(\lambda - \delta, \lambda + \delta) \subset (a, b)$, then from Wiener’ s generalized theorem (^{1}) and from 3) it follows that the equation

$$f * h = \mu_\delta(t) e^{it\lambda}$$

has a solution $h(t) \in L_\varphi$. Multiplying (1) by $h(t)$ and integrating with respect to t from $-\infty$ to ∞, after changing the order of integration we obtain:

$$\int_{-\infty}^{\infty} \mu_\delta(t) g(t) e^{it\lambda}\, dt = 0$$

or

$$\int_0^{\infty} \mu_\delta(t) g(t) e^{it\lambda}\, dt = -\int_{-\infty}^{0} \mu_\delta(t) g(t) e^{it\lambda}\, dt \qquad (a + \delta < \lambda < b - \delta). \quad (4)$$

The functions

$$G_\delta^+(z) = \int_0^{\infty} \mu_\delta(t) g(t) e^{itz}\, dt, \qquad \operatorname{Im} z \geq 0;$$

$$G_\delta^-(z) = -\int_{-\infty}^{0} \mu_\delta(t) g(t) e^{itz}\, dt, \qquad \operatorname{Im} z \leq 0, \qquad (5)$$

are analytic and bounded in their domains. Equality (4) means that $G_\delta^+(\lambda) = G_\delta^-(\lambda)$, $a+\delta < \lambda < b-\delta$, i.e. that the functions $G_\delta^+(z)$ and $G_\delta^-(z)$ form a single analytic function $G_\delta(z)$ in the plane cut along the rays $-\infty < x < a+\delta,\ b-\delta < x < \infty$.

We shall prove that the family $G_\delta(z)$, as $\delta \to 0$, converges in the interval $a < a' < x < b' < b$. To this end note that in the rectangle $a+\delta < x < b-\delta,\ |y| \le d$ the function $G_\delta(z)$ is analytic. We show that in the inner rectangle $D:\ a' < x < b',\ |y| \le d$, where $(a', b') \subset (a, b)$, the estimate

$$|G_\delta(z)| < M \quad \text{for all } \delta < \min(b-b',\, a'-a). \tag{6}$$

* See (^{3}), p. 127. Beurling (^{2}) calls this lemma Sjöberg' s lemma.

From the integral representation (5) of the function $G_\delta(z)$ we obtain, for $y > 0$,

$$|G_\delta(x+iy)| < K_g \int_0^\infty \varphi(t) e^{-yt}\, dt \le K_g M\left(\frac{y}{2}\right) + C.$$

An analogous inequality is obtained for $y < 0$.
By Lemma 1, $M_1(y) = K_g M(y/2) + C$ satisfies the condition

$$\int_0^{2k} \ln\ln M_1(y)\, dy < \infty.$$

This condition ensures the possibility of applying Lemma 2 and, thus, inequality (6) is satisfied. Since in the upper and lower half-planes $G_\delta^+(z)$ and $G_\delta^-(z)$ converge respectively to $G^+(z)$ and to $G^-(z)$, it follows by the well-known Stieltjes theorem that $G_\delta(z)$ converges in the rectangle D to some function $G(z)$, and

$$G(z) = G^+(z), \quad \operatorname{Im} z < 0; \qquad G(z) = G^-(z), \quad \operatorname{Im} z < 0.$$

Theorem 1 is proved. From it the following theorem is obtained.

Theorem 2*. *If $\varphi(x)$ satisfies the conditions of Theorem 1, $f(x) \in L_\varphi$, $g(x) \in M_\varphi$, and if the Fourier transform of the function $f(x)$ does not vanish outside the interval $(-h, h)$, then from equality (1) it follows that $g(x)$ coincides with a certain entire function of finite degree everywhere on the real axis except, possibly, on a set of measure zero. Moreover,*

$$|g(z)| < A\varphi(x) \exp h|y|. \tag{7}$$

Proof. The Laplace transform $G(z)$ of the function $g(x)$ is, by Theorem 1, an analytic function outside the interval $[-h', h]$, where $h' < h$. Take an interval

(a, b) not intersecting the segment $[-h', h']$, and choose the rectangle D as was done in the proof of Theorem 1. Since

$$|G(cz)| = |G(cx + icy)| \leq M_1(cy) \leq M_1(y)$$

for any $c > 1$, it follows that $|G(cx)| < M \ (a < x < b)$.

Thus, $G(z)$ is bounded at infinity. Since $G(z)$ is a Laplace transform, $G(z) \to 0$ as $|z| \to \infty$. Hence it follows that

$$g(x) = \frac{1}{2\pi i} \int_\Gamma e^{-ix\zeta} G(\zeta)\, d\zeta,$$

where Γ is a rectangle enclosing the segment $[-h', h']$, and, consequently, $g(z)$ is an entire function of finite degree, less than or equal to h'. By condition 3), in the whole plane the following holds^5:

$$\lg |g(z)| \leq \frac{|y|}{\pi} \int_{-\infty}^{\infty} \frac{\ln |g(t)|}{(t-x)^2 + y^2}\, dt + h'|y|.$$

From this inequality and condition 2), (7) is easily obtained.

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* This theorem, for the case $\varphi(x) = 1$, was proved by Mandelbrojt (^{4}).

Note: Figure translations are in progress. See original paper for figures.

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