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CHZHAN CHZHI-FEN

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Abstract

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MATHEMATICS

CHZHAN CHZHI-FEN

ON THE UNIQUENESS OF LIMIT CYCLES OF CERTAIN EQUATIONS OF NONLINEAR OSCILLATIONS

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Many authors have proved theorems on the existence of at least one limit cycle for certain equations of nonlinear oscillations. The question of the existence of a unique limit cycle is of great practical importance, but comparatively few results have been obtained in this direction.

In papers ($^{1-6}$) theorems have been proved on the existence of a unique limit cycle for a certain equation of nonlinear oscillations.

In the present note a number of theorems are established on the existence of a unique limit cycle for the system

$$\frac{dy}{dt} = g(x), \qquad \frac{dx}{dt} = -\varphi(y) - F(x). \tag{1}$$

Theorem 1. Consider the equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{2}$$

or the equivalent system

$$\frac{dy}{dt} = g(x), \qquad \frac{dx}{dt} = -y - F(x), \tag{3}$$

where

$$F(x) = \int_0^x f(x)\,dx.$$

If the following conditions are satisfied:

1) $xg(x) > 0, \quad x \neq 0; \quad G(-\infty) = G(\infty) = \infty$, where

$$G(x) = \int_0^x g(x)\,dx;$$

$g(x)$ is continuous and satisfies the Lipschitz condition on every finite interval;

2) $f(x)$ is continuous and $f(x)/g(x)$ is nondecreasing as x increases on the intervals $-\infty < x < 0,\ 0 < x < \infty$; $f(x)/g(x) \neq \text{const}$ in a neighborhood of $x = 0$,

then system (3) has no more than one limit cycle, and if this cycle exists, then it is stable.

The existence of limit cycles of system (3) in this case will be ensured if, in addition, one assumes that $f(x) > 0$ for sufficiently large $|x|$.

The idea of the proof is as follows. The point $(0, 0)$ is the only singular point for system (3); therefore all limit cycles in the (x, y)-plane are nested in one another and all surround the origin. Under conditions 1), 2) one can prove that either $(0, 0)$ is a repelling singular point, or system (3) has no limit cycles at all.

Suppose that system (3) has limit cycles. Since $(0, 0)$ is a repelling singular point, the limit cycle L_1 nearest to $(0, 0)$ cannot be unstable. Let $x = x_1(t)$, $y = y_1(t)$ be the equation of L_1. By the Poincaré criterion (5),

$$\oint_{L_1} f[x_1(t)]\,dt \geq 0.$$

If outside L_1 there exists another limit cycle L_2, then it is proved that

$$\oint_{L_2} f[x_2(t)]\,dt > \oint_{L_1} f[x_1(t)]\,dt,$$

i.e.

$$\oint_{L_2} f[x_2(t)]\,dt > 0,$$

where $x = x_2(t)$, $y = y_2(t)$ is the equation of the cycle L_2. Then, again by the Poincaré criterion, L_2 is stable. Consequently, all limit cycles lying outside L_1 are stable. Since two stable limit cycles cannot be adjacent, if L_1 is stable, then outside L_1 there are no limit cycles; but if L_1 is semistable, then outside L_1 there may, generally speaking, exist one more stable limit cycle. It is proved that the second case cannot occur, i.e. system (3) has no more than one limit cycle.

An analogous theorem is proved for system (1), if, in addition to conditions 1), 2) of Theorem 1, one requires additionally:

3) $y\varphi(y) > 0,\ y \neq 0;\ \varphi(\pm\infty) = \infty;\ \varphi(y)$ is continuous, strictly monotone and satisfies the Lipschitz condition; for $y = 0$ the function $\varphi(y)$ has right and left derivatives $\varphi'_+(0)$, $\varphi'_-(0)$;

4) $\varphi'_+(0) \cdot \varphi'_-(0) \neq 0$ in the case $f(0) = 0$, where $f(x) = F'(x)$.

By Theorem 1 the equation $\ddot{x} + (x^2 + \alpha x + \beta)\dot{x} + x = 0$ has a unique limit cycle for $\beta < 0$ for arbitrary α. The Van der Pol equation is a special case of this equation. The question of the number of limit cycles for this equation had not previously been completely resolved by known theorems.

Remark. If in Theorem 1 there exist $\bar{x}_1 > 0$ and $\bar{x}_2 < 0$ such that $xF(x) \leq 0$ for $\bar{x}_2 \leq x \leq \bar{x}_1$; $xF(x) > 0$ for $x > \bar{x}_1,\ x < \bar{x}_2$, and if $u(\bar{x}_1) \leq -u(\bar{x}_2)$ (or $u(\bar{x}_1) \geq -u(\bar{x}_2)$), where $u(x) = \sqrt{2G(x)}\operatorname{sgn} x$, then it is sufficient that condition 2) hold only on the intervals $-\infty < x < 0,\ \bar{x}_1 < x < \infty$ (or, respectively, on the intervals $-\infty < x < \bar{x}_2,\ 0 \leq x < \infty$), and that on the interval $0 < x < \bar{x}_1$ (or, respectively, on the interval $\bar{x}_2 < x < 0$) one have $f(x)/g(x) \leq f(\bar{x}_1)/g(\bar{x}_1)$ or, respectively, $f(\bar{x}_2)/g(\bar{x}_2) \geq f(x)/g(x)$.

Theorem 2. The equation (2), or the equivalent system (3), is considered.

If condition 1) of Theorem 1 holds and if the following condition is satisfied:

2′) $f(x)$ is continuous and $F[x(u)]/u$ does not decrease as $|u|$ increases, where $x = x(u)$ is determined by the equation $u = u(x) = \sqrt{2G(x)}\operatorname{sgn} x$,

then system (3) has no more than one limit cycle.

This assertion is a generalization of Conti’ s theorem (^{4}), in which, in addition to these conditions, there is also the condition $|F(x(u))/u| < 2$.

If, in addition to the conditions of Theorem 2, one also has $xF(x) < 0$ for $0 < |x| < \delta_1$, $xF(x) > 0$ for $|x| > \delta_2$, where $\delta_1 < \delta_2$, then system (3) has a unique limit cycle.

This assertion is a generalization of Massera’ s theorem (^{3}), which was also proved by M. G. Khudai-Verenov (^{6}), and still more of J. Stoker’ s theorem (^{5}). In Massera’ s theorem equation (2) is considered for $g(x) = x$, and instead of condition 2′ it is assumed that $f(x)$ is continuous, $f(x) > 0$ for $x > x_1 > 0$, $x < x_2 < 0$; $f(x) < 0$ for $x_2 < x < x_1$; $f(x)$ does not decrease as $|x|$ increases.

Theorem 3*. *Consider equation (2), or the equivalent system (3), for $g(x) = x$.*

If the following conditions are satisfied:

1) $f(x)$ is continuous; $f(x) < 0$ for $x_2 < x < x_1$; $f(x) > 0$ for $x > x_1$, $x < x_2$, where $x_1 > 0$, $x_2 < 0$;

2) $f(x)$ does not decrease as $|x|$ increases on the intervals $x_1 < x < \infty$ (or $0 < x < \infty$) and $-\infty < x < 0$ (or, respectively, $-\infty < x < x_2$), if

$x_1 \leq -x_2$ (or, respectively, if $x_1 \geq -x_2$),

then system (3) has a unique limit cycle, which is stable.

In the case $x_1 = -x_2$, by Sansone' s theorem (^{2}), condition 2) can be replaced by a weaker condition.

If condition 2) is replaced by the statement that $f(x)$ does not decrease as $|x|$ increases on the intervals $x_1 < x < \infty$, $-\infty < x < x_2$, then system (3) has no more than two limit cycles containing the points $(x_1, 0)$, $(x_2, 0)$ in their interior.

Theorem 4. *Consider equation (2) for $g(x) = x$, or the equivalent system*

$$\begin{aligned} \frac{dx}{dt} &= v, \\ \frac{dv}{dt} &= -x - f(x)v. \end{aligned} \tag{4}$$

Here

$$f(x) = a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \cdots + a_2x^2 + a_1x + a_0.$$

System (4) has no more than one limit cycle in the region R, bounded by the circles $x^2 + v^2 = \delta_2^2$ and $x^2 + v^2 = \delta_1^2$, $\delta_2 > \delta_1$, if

$$L_1(\delta_1, \delta_2) = a_{2k}\lambda_{2k} + a_{2k-1}\lambda_{2k-1} + \cdots + a_2\lambda_2 + a_0 > 0$$

or

$$L_2(\delta_1, \delta_2) = a_{2k}\gamma_{2k} + a_{2k-1}\gamma_{2k-1} + \cdots + a_2\gamma_2 + a_0 < 0,$$

where

$$\lambda_{2l} = \begin{cases} \dfrac{(2l-1)!!}{(2l)!!}\,\delta_1^{2l}, & a_{2l} > 0, \\ 0, & a_{2l} = 0, \quad l = 1, 2, \dots, k, \\ \dfrac{(2l-1)!!}{(2l)!!}\,\delta_2^{2l}, & a_{2l} < 0, \end{cases}$$

$$\lambda_{2l+1} = \begin{cases} \dfrac{\sqrt{(4l+1)!!}}{\sqrt{(4l+2)!!}}\,\delta_2^{2l+1}, & a_{2l+1} > 0, \\ 0, & a_{2l+1} = 0, \quad l = 1, 2, \dots, k-1, \\ \dfrac{\sqrt{(4l+1)!!}}{\sqrt{(4l+2)!!}}\,\delta_2^{2l+1}, & a_{2l+1} < 0; \end{cases}$$

* A generalization of Massera’ s theorem (3) in another direction.

$$\gamma_{2l} = \begin{cases} \dfrac{(2l-1)!!}{(2l)!!}\,\delta_2^{2l}, & a_{2l} > 0, \\ 0, & a_{2l} = 0, \\ \dfrac{(2l-1)!!}{(2l)!!}\,\delta_1^{2l}, & a_{2l} < 0, \end{cases} \qquad l = 1, 2, \dots, k;$$

$$\gamma_{2l+1} = \begin{cases} \dfrac{\sqrt{(4l+1)!!}}{\sqrt{(4l+2)!!}}\,\delta_2^{2l+1}, & a_{2l+1} > 0, \\ 0, & a_{2l+1} = 0, \\ -\dfrac{\sqrt{(4l+1)!!}}{\sqrt{(4l+2)!!}}\,\delta_2^{2l+1}, & a_{2l+1} < 0, \end{cases} \qquad l = 1, 2, \dots, k-1.$$

In the proof, the equality

$$\oint_L x^n(t)\,dt = \oint_L (n-1)x^{n-2}(t)\dot{x}^2(t)\,dt,$$

where $x = x(t)$ is an equation of the periodic solution of equation (2) for $g(x) = x$, is used essentially.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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