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Abstract

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A NEW SPACE OF CLOSED SETS AND MULTIVALUED MAPPINGS OF BICOMPACTA

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Open mappings f of a bicompactum X onto a bicompactum Y, as shown in my paper (1), are characterized by the fact that the set of elements of the corresponding decomposition of the space X is a closed set in the space ψX of closed sets of the bicompactum X, defined as usual (see, for example, (1), § 3). The question arises of an analogous characterization of arbitrary continuous mappings of bicompacta. It is easy to understand that the space ψX is unsuitable for this purpose. In this note we define a new space of closed sets, which not only immediately solves the problem posed, but in general is closely connected with the theory of continuous—not only single-valued, but also multivalued—mappings of bicompacta.

1. The space $\varkappa X$ and its properties. Let X be an arbitrary T_1-space. The points of the space $\varkappa X$ are all nonempty closed subsets of the space X. To define an arbitrary neighborhood $O(F_0)$ of an arbitrary point $F_0 \in \varkappa X$, take an arbitrary neighborhood OF_0 of the set F_0 in the space X and define $O(F_0)$ as the collection of all closed subsets of the space X lying in OF_0.

It is easy to see that the neighborhood space $\varkappa X$ thus defined is a T_0-space, topologically containing the space X; in what follows we shall call the points $x \in X$ "small" points of the space $\varkappa X$.

The space $\varkappa X$ is never Hausdorff, nor even a T_1-space, except in the trivial case when X consists of a single point.

1°. The identity mapping of the space ψX onto $\varkappa X$ is continuous.

2°. The condition $F_1 \subseteq F_0 \subseteq X$ is necessary and sufficient in order that, in the space $\varkappa X$, the point F_0 be contained in the closure of the point F_1.

Hence, in turn, it follows that:

3°. The small points of the space $\varkappa X$ are characterized by the fact that no small point is contained in the closure of any point of the space $\varkappa X$ distinct from it.

4°. Under any topological mapping of the space $\varkappa X$ onto itself or onto a space $\varkappa Y$ (where Y is some T_1-space), every small point goes into a small one, and every large (i.e. non-small) point goes into a large one. The "largest" point $X \in \varkappa X$ goes into itself under every continuous mapping of the space $\varkappa X$ onto itself.

Remark. For the space ψX, proposition 4° loses its force (already when X is the Cantor perfect set).

A consequence of proposition 4° is:

5°. The spaces $\varkappa X$ and $\varkappa Y$ are homeomorphic if and only if the spaces X and Y are homeomorphic.

6°. All nonempty closed sets of the space $\varkappa X$ contain the "largest" point X of the space $\varkappa X$.

Therefore:

7°. Whatever the T_1-space X may be, the space $\varkappa X$ is connected and bicompact.

8°. If X is bicompact, then the space $\varkappa X$ has the "fixed-point property,"i.e. under every single-valued continuous mapping of the space $\varkappa X$ into itself there is at least one fixed point.

Remark. As the already considered example of Cantor' s perfect set shows, the space ψX, generally speaking, does not have the fixed-point property.

Let τ be some infinite cardinal number. As usual, let D^τ denote the Cantor discontinuum of weight τ, i.e. the topological product of τ copies of the space consisting of two isolated points. It turns out that:

9°. The space $\varkappa D^\tau$ contains a topological image of every T_0-space of weight $\leqslant \tau$.

2. Let X be a bicompactum. It is easy to see that a decomposition of the space X into disjoint closed sets is continuous in the sense of P. S. Aleksandrov (see, for example, (^{2}), p. 42) if and only if the set Z of all elements of this decomposition is bicompact in the space $\varkappa X$ (the set Z is never closed in $\varkappa X$!)—this also answers the question posed at the beginning of the note.

We pass to applications of the space $\varkappa X$ to the study of multivalued continuous mappings of bicompacta.

3. **Multivalued continuous mappings.** We shall consider only such multivalued mappings f of a space X into a space Y which consist in assigning to each point $x \in X$ a closed set fx of the space Y ("the image of the point x"). If M is some set lying in X, then by its image we mean the sum of the images of the points comprising the set M:

$$fM = \bigcup_{x \in M} fx.$$

Definition. A mapping f of a space X into a space Y is called **continuous at the point** x, if for every neighborhood Ofx of the set fx in Y one can find a neighborhood Ox of the point x such that $fOx \subseteq Ofx$. If the mapping f is continuous at all points $x \in X$, then it is called **continuous on the whole space** X.

In what follows we shall consider only continuous mappings of a bicompactum X onto a bicompactum Y. In this case the set

$$f'y = \mathfrak{E}(x,\ fx \ni y) \tag{1}$$

(i.e. the set of all $x \in X$ for which $y \in f(x)$) is closed in X, so that the continuous mapping f of the bicompactum X onto Y defines, by means of equality (1), a mapping f' of the bicompactum Y onto X. This mapping is continuous and is called the **inverse to the mapping** f, which is justified by the fact that the inverse mapping to the mapping f' is the mapping f,

$$(f')' = f.$$

Theorem 1. *In order that a mapping f of a bicompactum X onto a bicompactum Y be continuous, it is necessary and sufficient that it be closed (i.e. fA is closed in Y for every A closed in X) and that the sets $f'y$ be closed for all $y \in Y$.*

4. **Continuous mappings and the spaces $\varkappa X, \varkappa Y$.** Considering the sets fx as points of the space $\varkappa Y$, we see that every closed mapping f of a space X into a space Y gives rise to a single-valued mapping f^* of the space $\varkappa X$ into $\varkappa Y$. It is easy to see that a mapping f of a bicompactum X onto a bicompactum Y is continuous if and only if

only when the mapping f^* of the space $\varkappa X$ into the space $\varkappa Y$ is continuous.

The following theorem holds, which is an analogue of the assertion stated in § 2, namely:

Theorem 2. *A mapping f of a bicompactum X onto a bicompactum Y is continuous if and only if the sets $f'y$ are closed in X for all $y \in Y$ and, whatever closed set B in Y is taken, the set of all $f'y$ for $y \in B$ is bicompact in the topology of the space $\varkappa X$.*

In this theorem one may replace the closed set B in Y by a closed set A in X and then require that the set f^*A be closed in $\varkappa Y$.

5. Convergence of sequences over a directed set. Let a directed set Θ be given. If to each element $\alpha \in \Theta$ there is assigned some $x_\alpha \in X$ or some $M_\alpha \subseteq X$, then we shall speak of a sequence of points x_α (or of sets M_α) "over the directed set Θ." We shall say that the sequence $\{M_\alpha\}$ converges in the space X to the **closed set** F, if for every neighborhood OF of the set F in the space X there exists an $\alpha_0 \in \Theta$ such that for all $\alpha > \alpha_0$ we have $M_\alpha \subseteq OF$. In particular, we obtain the definition of convergence of a sequence of points to a point or to a closed set. Obviously, the convergence of a sequence of closed sets F_α to a set F is equivalent to the convergence of the corresponding sequence of points F_α

of the space $\varkappa X$ to the point F_0. In the sense of this convergence, one and the same sequence of points or sets may have many limiting sets; however, if X is a bicompactum, then among all (closed) limiting sets of a given sequence of sets there is one and only one minimal one.

It is easy to prove:

Theorem 3. *A mapping f of a bicompactum X onto a bicompactum Y is continuous if and only if, whatever sequence of points $\{x_\alpha\}$ converging in X to a point x_0 is taken, the sequence of closed sets fx_α converges in Y to the set fx_0.*

6. The graph of a mapping. The graph of a mapping f of a space X into a space Y is the set lying in the topological product $X \times Y$ of all points (x, y) such that $x \in X$, $y \in fx$.

From the theorems formulated above there easily follows one more characterization of continuous mappings:

Theorem 4. *A mapping f of a bicompactum X into a bicompactum Y is continuous if and only if the graph of this mapping is closed in the space $X \times Y$.*

7. Strongly continuous and open mappings. Let f be some mapping of a space X onto a space Y. We shall call the **large inverse image** of a set $B \subseteq Y$ under the mapping f the set $f'B \subseteq X$, i.e. the set of all those points $x \in X$ for which the set $fx \cap B$ is nonempty. Similarly, we shall call the **small inverse image** of a set B the set $f^{\#}B$ of all points $x \in X$ for which $fx \subseteq B$. Obviously, $f^{\#}B = X \setminus f'(Y \setminus B)$. We can now say that continuous mappings are those mappings for which the small inverse image of every open (in Y) set is open in X, or, equivalently, the large inverse image of every closed set is closed. It is natural to call **strongly continuous** those mappings for which both the large and the small inverse images of every open set are open (or, equivalently, the large and small inverse images of every closed set are closed). These are the mappings which W. Strother (^{3}) calls continuous. Obviously, the class of strongly continuous mappings coincides with the class of mappings inverse to open ones.

A closed mapping f (of a bicompactum X into a bicompactum Y) generates

in a natural way a single-valued mapping f^* of the space ψX into ψY. As Strother proves (^{3}), the strong continuity of the mapping f is equivalent to the continuity of the mapping f^*. Strong continuity can also be characterized by means of convergence—in the same way as was done in Theorem 3, except that convergence of sets is now understood as convergence of points in the space ψY.

8. The following propositions hold, extending to the case of multivalued mappings known results concerning single-valued mappings.

Theorem 5. *Let f be a strongly continuous mapping of a connected space X onto a space Y. If, for at least one point $x_0 \in X$, the set fx_0 is connected, then the entire space Y is connected.*

Theorem 6. *Let f be a (multivalued) open mapping of a bicompactum X onto a bicompactum Y. Let C be a connected closed set in Y, and let k be any component of the set $A = f'C$. Then $fk \supset C$.*

Theorem 7. *Keeping the notation of the preceding theorem, the set $C \cap \delta(y, f'y \subset k)$—the "small image in C of the component k" —is either empty or coincides with all of C.*

Simple examples show that both of these possibilities can in fact occur.

Theorem 8. *Let f be an open mapping, together with its inverse f', of a bicompactum X onto a bicompactum Y, and let $C \subseteq Y$ be a connected set, closed in Y, containing interior points. Then the set $A = f'C$ consists of a finite number of components.*

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[3] W. Strother, *Duke Math. J.*, **22**, No. 4, 551 (1955).

Note: Figure translations are in progress. See original paper for figures.