MATHEMATICS

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Abstract

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ON HOMOGENEOUS DIFFERENCE SCHEMES

In the paper ([1]) the problem was posed of finding difference schemes suitable for a uniform solution of differential equations in as broad a class of coefficients as possible. The present work is a further development of ([1]).

§ 1. Consider the equation

$$Lu = -f(x) \qquad (0 < x < 1), \tag{1}$$

where L is some linear differential operator.

Let $S_N(x_0 = 0,\ x_1 = h, \dots, x_i = ih, \dots, x_N = Nh = 1)$ be a difference grid;

$$L_h y_i^h = -F_i^h \tag{2}$$

is the difference equation corresponding to equation (1).

The linear difference operator L_h is defined by means of the matrix of coefficients a_{ij}^h of the system of linear equations (2), which are functions of the step of the difference grid $h = \frac{1}{N}$. To obtain the difference equations (2), it is also necessary to specify the functionals $F_i^h[f]$, defined in some class $\{f(x)\}$, and the boundary conditions.

§ 2. Consider the class of equations

$$L^{(p(x))}u = -f(x). \tag{3}$$

We shall call the functions $(p(x)) = \{p_1(x), p_2(x), \dots, p_m(x)\}$ the coefficients of equation (3).

The class of differential equations (3) will be defined if the type of the operator L is fixed and the class to which the coefficients $(p(x))$ belong is specified.

Let $L_h^{(p)}$ denote the class of difference operators L_h whose matrix elements a_{ij}^h are functionals defined in the class of coefficients $(p(x))$ under consideration and depending on the parameter h. Such a functional matrix $L_h^{(p)} = (a_{ij}^h[p(x)])$ will be called a difference scheme.

§ 3. We introduce the definitions needed in what follows:

1. We shall call $C_m^\gamma(f)$ the class of functions $f(x)$ having an m-th derivative satisfying on the interval $[0,1]$ a Hölder condition of order $\gamma > 0$. If the m-th derivative is continuous, then we shall denote the corresponding class of functions by $C_m(f)$. In particular, $C_0(f)$ is the class of continuous functions.

2. We shall say that $f(x)$ belongs to the class $Q_m(f)$ if $f(x)$ and its m derivatives are piecewise continuous on $(0,1)$. If, in addition, the m-th derivative in each of the intervals of continuity satisfies a Hölder condition of order γ, then the corresponding class will be called Q_m^γ. In particular, Q_0 is the class of piecewise-continuous functions.

3. Let $u(x)$ be some solution of the equation $Lu = -f$; let y_i^h be the corresponding solution of the equation $L_h y_i^h = -F_i^h$; $z_i^h = y_i^h - u(x_i)$; $z(x,h)$ a function equal to z_i^h for $x = x_i$ and linear between neighboring nodal points of the grid. We shall say that the difference operator L_h converges to the differential operator L if the function $z(x,h)$ tends uniformly to zero as $h \to 0$ for an arbitrary function $f(x)$ from some class, i.e.

$$|z(x,h)| < \rho(h), \qquad \text{where } \rho(h) \to 0 \text{ as } h \to 0.$$

If $z(x,h) = O(h^n)$ or $|z(x,h)| < Mh^n$, where M is a positive constant depending on the choice of the function $f(x)$, then we shall say that L_h has n-th (integral) order of accuracy relative to L.

4. The difference operator L_h has n-th order of approximation relative to the operator L if there is an m such that, for any function $y(x)$ of class C_m, for all values of N and at all points of the difference grid we have

$$|L_h y_i - (Ly)_i| < Mh^n,$$

where M is a positive constant depending on the choice of $y(x)$. Analogously, one may speak of the order of approximation on some interval $[a,b] \subset [0,1]$.

5. If, for any choice of the coefficients $(p(x))$ from a given functional class, the difference scheme gives a difference operator L_h converging to the operator L which corresponds to the chosen coefficients $(p(x))$, then the difference scheme $L_h^{(p)}$ will be called convergent in the given class of coefficients. Similarly, we shall say that the difference scheme $L_h^{(p)}$ has n-th integral order of accuracy (or n-th order of approximation) in the given class of coefficients if, for any functions $(p(x))$ from this class, the difference operator L_h has n-th integral order of accuracy (n-th order of approximation).

6. The difference schemes $L_h^{(p)}$ and $\overline{L}_h^{(p)}$ are equivalent in the sense of convergence in some class of coefficients $(p(x))$ if, for any functions from this class, the difference $y(x,h) - \overline{y}(x,h)$ tends uniformly to zero as $h \to 0$.

If $y(x,h) - \overline{y}(x,h) = O(h^n)$ (or $L_h^{(p)} y_i - \overline{L}_h^{(p)} y_i = O(h^n)$) for any function $p(x)$ from the given class, then the difference schemes $L_h^{(p)}$ and $\overline{L}_h^{(p)}$ have n-th integral (or local) order of equivalence.

It is obvious that:

If $L_h^{(p)}$ and $\overline{L}_h^{(p)}$ have n-th order of accuracy, then they have n-th integral order of equivalence.

If $L_h^{(p)}$ and $\overline{L}_h^{(p)}$ have n-th integral (or local) order of equivalence and $L_h^{(p)}$ has n-th order of accuracy (or n-th order of approximation), then $\overline{L}_h^{(p)}$ possesses the same property.

7. We shall call a difference scheme

$$L_h^{(p)} = (a_{ij}^h[p(x)])$$

a symmetric scheme if the difference operator L_h remains unchanged under a change in the direction of the x-axis. The symmetry conditions have the form:

1) $a_{ij}^h[p(x)] = a_{i,\,2i-j}^h[p(2x_i + x)]$ $(x_i = ih)$;
2) $0 \leq (2i-j)h \leq 1$.

8. The difference scheme $L_h^{(p)}$ is called a homogeneous scheme if the elements a_{ij}^h of the matrix L_h at all points i are determined uniformly for all functions $(p(x))$, i.e. are functionals of the form

$$a_{ij}^h[p(x)] = \overline{a}_{j-i}^h[\overline{p}(s)], \qquad \overline{p}(s) = p(x_i + sh), \qquad -n_1 \leq j - i \leq n_2.$$

If a homogeneous scheme is symmetric, then

1) $n_1 = n_2$;
2) $a_{j-i}^h[p(x_i + sh)] = a_{i-j}^h[p(x_i - sh)]$.

§ 4. Consider on the interval $0 \leq x \leq 1$ the first boundary-value problem for the class of equations

$$L^{(p)} u = \frac{d}{dx}\left[\frac{1}{p(x)}\frac{du}{dx}\right] = -f(x) \qquad (0 < M_1 \leq p(x) \leq M_2). \tag{4}$$

Let

$$L_h^{(p)} y_i = \frac{1}{h^2}\left[\frac{1}{A_i^h}\, y_{i-1} + \frac{1}{C_i^h}\, y_i + \frac{1}{B_i^h}\, y_{i+1}\right] \tag{5}$$

be a three-point homogeneous difference scheme whose coefficients are

$$A_i^h = A^h[p(x_i + sh)], \qquad B_i^h = B^h[p(x_i + sh)], \qquad C_i^h = C^h[p(x_i + sh)],$$

where $A^h[\bar{p}(s)]$, $B^h[\bar{p}(s)]$, $C^h[\bar{p}(s)]$ are functionals of the function $\bar{p}(s)$, specified for $-1 < s < 1$.

In order that the difference scheme have, in the class $C_m(p)$ $(m \geq k+1,\ k = 1, 2)$, k-th order of approximation, it is necessary and sufficient that the conditions

$$\frac{1}{A_i^h} + \frac{1}{C_i^h} + \frac{1}{B_i^h} = O(h^{k+2}), \tag{6}$$

$$\frac{1}{h}\left[\frac{1}{B_i^h} - \frac{1}{A_i^h}\right] = -\frac{p_i'}{p_i^2} + O(h^k), \qquad \frac{1}{2}\left(\frac{1}{B_i^h} + \frac{1}{A_i^h}\right) = \frac{1}{p_i} + O(h^k). \tag{7}$$

Lemma 1. If the difference scheme (5) has k-th order of approximation, then the scheme

$$L_h^{(p)} y_i = \frac{1}{h^2}\left[\frac{1}{B_i^h}(y_{i+1} - y_i) - \frac{1}{A_i^h}(y_i - y_{i-1})\right] \tag{8}$$

has the same property.

§ 5. The homogeneous difference scheme (8) will be called p-linear (or simply linear) if: 1) $A^h[\bar{p}]$ and $B^h[\bar{p}]$ are linear regular functionals (*); 2) for $0 \leq h \leq h_0 < 1$ the representation

$$\begin{aligned} A^h[\bar{p}] &= A^{(0)}[\bar{p}] + hA^{(1)}[\bar{p}] + h^2 A^{(2)}[\bar{p}] + O_{\bar{p}}(h^3), \\ B^h[\bar{p}] &= B^{(0)}[\bar{p}] + hB^{(1)}[\bar{p}] + h^2 B^{(2)}[\bar{p}] + O_{\bar{p}}(h^3), \end{aligned} \tag{9}$$

holds, where $|O_{\bar{p}}(h^3)| \leq K_{\bar{p}} h^3$; $K_{\bar{p}}$ is a constant depending on the choice of $\bar{p}$, and all coefficients of the powers h^0, h, h^2 are linear regular functionals.

The linear difference scheme

$$L_h^{(p)} y_i = \frac{1}{h^2}\left[\frac{1}{B_i}(y_{i+1} - y_i) - \frac{1}{A_i}(y_i - y_{i-1})\right],$$

$$A_i = A[p(x_i + sh)], \qquad B_i = B[p(x_i + sh)] \tag{10}$$

is called canonical if the functionals $A[\bar{p}(s)]$ and $B[\bar{p}(s)]$ do not depend on h.

Lemma 2. If a linear difference scheme of the form (8) has k-th order of approximation ($k = 1, 2$), then the corresponding canonical scheme, for which $A = A^{(0)}$, $B = B^{(0)}$, also has k-th order of approximation.

We note that, for a scheme of first order of approximation, the conditions

$$A[1] = 1, \qquad B[1] = 1, \qquad B[s] - A[s] = 1,$$

must be satisfied, while for a scheme of second order of approximation the conditions are

$$A[1] = 1, \qquad B[1] = 1, \qquad A[s] = -0.5, \qquad B[s] = 0.5, \qquad A[s^2] = B[s^2].$$

Lemma 3. *If a canonical scheme of first order of approximation is symmetric, then it has second order of approximation.*

§ 6. The requirement that $L_h^{(p)}$ be defined in $Q_m(p)$ means that $A_i \neq 0$, $B_i \neq 0$ at no point of the difference grid for any function $p \in Q_m$. These conditions will be fulfilled if the functionals A and B are positive $\big(A[\bar{p}] > 0,\ B[\bar{p}] > 0$ for $\bar{p}(s) > 0\big)$ (see (2)).

If the canonical scheme $L_h^{(p)}$ is symmetric and the functionals $A[\bar{p}(s)]$, $B[\bar{p}(s)]$ are positive, then such a difference scheme is called **normal**. In what follows we shall consider normal schemes.

The relation between the order of approximation and the order of accuracy is established by the following theorem:

Theorem. *Convergence of a normal difference scheme in the sense of approximation is necessary and sufficient for integral convergence; more precisely:*

1) *If a normal scheme converges in $C_1'(p)$, then it has first order of approximation in $C_2(p)$ and, by virtue of symmetry, second order of approximation for $p(x) \in C_3$.*
2) *If a normal scheme has second order of approximation in $C_3(p)$, then it converges in $C_1'(p)$, has first order of accuracy in $C_2(p)$, and second order of accuracy in $C_3(p)$.*

Questions concerning convergence and the order of accuracy of normal difference schemes in the class $Q_m(p)$ will be considered separately.

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[1] A. N. Tikhonov, A. A. Samarskii, *DAN*, **108**, No. 3 (1956).
[2] A. N. Tikhonov, A. A. Samarskii, *DAN*, **122**, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.