

# ON SOME PROPERTIES OF AXISYMMETRIC SUPERSONIC GAS FLOWS

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**Abstract**

**Full Text**

**HYDROMECHANICS**

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**ON SOME PROPERTIES OF AXISYMMETRIC SUPERSONIC GAS FLOWS**

*(Presented by Academician A. A. Dorodnitsyn, 3 VI 1958)*

Let us consider supersonic axisymmetric gas flows determined by a prescribed characteristic of the first family  $AC$  and by the generator  $AB$  of the body of revolution being streamlined (Fig. 1). Let the gas move from point  $A$  to point  $B$ . Such gas flows obey the canonical system of equations (1)

$$\begin{aligned}
 r_\eta &= \operatorname{tg}(\vartheta + \alpha)x_\eta, & r_\xi &= \operatorname{tg}(\vartheta - \alpha)x_\xi, \\
 \vartheta_\eta - f_\eta + \frac{\sin \vartheta \sin \alpha}{\sin(\vartheta + \alpha)} \frac{r_\eta}{r} - \sin 2\alpha \cdot \varphi_\eta &= 0, & (1) \\
 \vartheta_\xi + f_\xi - \frac{\sin \vartheta \sin \alpha}{\sin(\vartheta - \alpha)} \frac{r_\xi}{r} + \sin 2\alpha \cdot \varphi_\xi &= 0,
 \end{aligned}$$

where  $\xi, \eta$  are characteristic variables, constant respectively along characteristics of the first and second families;  $x, r$  are Cartesian coordinates in the meridional plane of the flow;  $\alpha$  is the Mach angle;  $\vartheta$  is the angle of inclination of the velocity to the axis of the flow;  $f$  is a function of  $\alpha$ , with

$$df = -\frac{1 + \cos 2\alpha}{\varkappa - \cos 2\alpha} d\alpha;$$

$\varkappa$  is the adiabatic exponent;

$$\varphi = \frac{\ln p - \varkappa \ln \rho}{2\varkappa(\varkappa - 1)}$$

is the entropy function, depending only on the stream function  $\psi$ ;  $p$  is the gas pressure;  $\rho$  is the density. The stream function is defined by the relation

$$d\psi = \frac{\omega}{\sin \alpha} (\cos \vartheta dr - \sin \vartheta dx),$$

where

Fig. 1

Figure 1: Fig. 1

$$\omega = \sqrt{\frac{\chi + 1}{2}} \rho_0 r \left( \frac{1 - \cos 2\alpha}{\chi - \cos 2\alpha} \right)^{\frac{1}{2} \frac{\chi + 1}{\chi - 1}};$$

$\rho_0$  is the density of the adiabatically decelerated gas.

**Fig. 1**

Let us compute the derivative  $\bar{f} = df/ds$  along an arbitrary characteristic of the second family at its point of intersection with  $AC$ , where  $s$  is the distance along the line  $\eta = \text{const}$ . Let  $s$  increase when moving downstream. The symbol  $d/ds$  will have the same meaning in what follows. We have

$$\bar{f} = \frac{f_\xi}{r_\xi} \sin(\vartheta - \alpha).$$

Find the partial derivative of  $\bar{f}$  with respect to  $\eta$ :

$$\bar{f}_\eta = \frac{f_{\xi\eta}}{r_\xi} \sin(\vartheta - \alpha) - \frac{f_\xi r_{\xi\eta}}{r_\xi^2} \sin(\vartheta - \alpha) + \frac{f_\xi}{r_\xi} \cos(\vartheta - \alpha) \cdot (\vartheta_\eta - \alpha_\eta). \quad (2)$$

The expressions for  $f_{\xi\eta}$  and  $r_{\xi\eta}$  entering the last equality are found from the system of equations (1). Substituting these expressions into (2) and choosing  $\eta = \psi$  on the characteristic  $AC$ , we obtain

$$\bar{f}_\eta = -\Phi_1 \bar{f}^2 - \Phi_2 \bar{f} - \Phi_3, \quad (3)$$

where

$$\Phi_1 = \frac{\chi + 1}{\omega \sin 2\alpha (1 + \cos 2\alpha)},$$

$$\Phi_2 = \frac{\chi + 1}{1 + \cos 2\alpha} f_\eta - (2 + \chi - 2 \cos 2\alpha) \frac{d\varphi}{d\psi} - \frac{3 \sin \vartheta \cos \alpha - \cos \vartheta \sin \alpha}{2\omega r},$$

$$\Phi_3 = \frac{\sin^2 \vartheta \sin 2\alpha}{\omega r^2} - \frac{\sin(\vartheta + \alpha)}{2r} f_\eta + \omega \sin 2\alpha \left[ 2 \sin^2 \alpha \left( \frac{d\varphi}{d\psi} \right)^2 - \frac{d^2 \varphi}{d\psi^2} \right]$$

$$- \left[ \frac{\sin(\vartheta + \alpha) \sin 2\alpha}{r} + \omega(\chi + 1 + \cos 2\alpha) f_\eta \right] \frac{d\varphi}{d\psi}.$$

Let us denote the derivative  $ds/df$  along the line  $\eta = \text{const}$  by  $\gamma$ . Taking into account that  $\bar{f} = 1/\gamma$ , from (3) we find the equation for  $\gamma$

$$\frac{d\gamma}{d\eta} = \Phi_1 + \Phi_2\gamma + \Phi_3\gamma^2, \quad (4)$$

where  $d/d\eta$  denotes the total derivative along a characteristic of the first family.

We establish the boundary condition for the function  $\gamma$  at the point  $A$ . Let the equation of the generator  $AB$  be  $r = F(x)$  (the angle of the tangent to  $AB$  coincides with the angle  $\vartheta$ ).

From the equalities for the total derivatives along  $AB$  (defining  $\xi$  so that  $\xi = x$  on  $AB$ ) we obtain at the point  $A$

$$x_\xi + x_\eta \frac{d\eta}{d\xi} = 1, \quad r_\xi + r_\eta \frac{d\eta}{d\xi} = F', \quad \vartheta_\xi + \vartheta_\eta \frac{d\eta}{d\xi} = F'' \cos^2 \vartheta.$$

Adjoining equations (1) to the last equalities and carrying out the calculations, we obtain the boundary condition for  $\eta = \psi_A$

$$\gamma = S\mathfrak{R}, \quad (5)$$

where

$$S = \frac{1}{\mathfrak{R}\omega(f_\eta + 2 \sin 2\alpha d\varphi/d\psi) - 2 \cos \alpha},$$

$\mathfrak{R}$  is the radius of curvature of the generator  $AB$  at the point  $A$  when approaching the point  $A$  from the side of the point  $B$ .

We now determine the character of the dependence of  $\gamma$  on  $\mathfrak{R}$ . To this end, differentiate equation (4) and the boundary condition (5) with respect to  $\mathfrak{R}$ , taking into account that in the equalities (4) and (5) only the quantity  $\gamma$  depends on  $\mathfrak{R}$ . We obtain that  $\delta\gamma$  is determined by the differential equation

$$\frac{d\delta\gamma}{d\eta} = (\Phi_2 + 2\Phi_3\gamma) \delta\gamma$$

and by the boundary condition for  $\eta = \psi_A$

$$\delta\gamma = -2S^2 \cos \alpha \cdot \delta\mathfrak{R}.$$

Integration gives

$$\delta\gamma = - \left[ 2S_A^2 \cos \alpha_A \exp \int_{\eta=\psi_A}^{\eta} (\Phi_2 + 2\Phi_3\gamma) d\eta \right] \delta\mathfrak{R}.$$

In what follows we shall assume that the functions  $f_\eta$ ,  $f_\eta d\varphi/d\psi$ ,  $(d\varphi/d\psi)^2$ ,  $d^2\varphi/d\psi^2$  are integrable on  $AC$ .

For  $r > 0$ ,  $0 < \alpha < \pi/2$  we can conclude that  $\delta\gamma < 0$  for  $\delta\mathfrak{R} > 0$ , or that  $\delta\alpha/ds < 0$  for  $\delta\mathfrak{R} > 0$ . The last equation of system (1) shows that,

that is,  $\delta d\vartheta/ds < 0$  for  $\delta\mathfrak{R} > 0$ . Thus, the following property holds.

**Property 1.** In axisymmetric supersonic flow past a body of revolution, an increase in the radius of curvature  $\mathfrak{R}$  of the generatrix  $AB$  of this body at the point  $A$  leads to a decrease in the derivatives  $d\alpha/ds$  and  $d\vartheta/ds$  on the specified characteristic  $AC$  (with signs taken into account).

It follows from Property 1 that, when the curvature of  $AB$  at the point  $A$  is equal to  $-\infty$ , the derivatives  $d\alpha/ds$  and  $d\vartheta/ds$  have minimum values.

Let, further, the characteristic  $AC$  intersect the shock wave  $DE$  at the point  $C$ , while ahead of the shock wave there is a uniform gas flow with velocity equal to  $V$  (the velocity is referred to the critical flow speed). The quantities  $\alpha, \vartheta, p, \rho$  behind the shock wave, for a fixed value of  $V$ , are determined from the known relations

$$\alpha = A(\sigma), \quad \vartheta = \Theta(\sigma), \quad p = P(\sigma), \quad \rho = R(\sigma),$$

where  $\sigma$  is the angle of inclination of the shock wave to the flow axis.

The derivative  $d\vartheta/ds$ , taken along the characteristic  $CG$  at the point  $C$ , is equal to

$$\frac{d\vartheta}{ds} = \frac{\sin \vartheta \sin^2 \alpha \cos(\sigma - \vartheta)}{r \sin(\vartheta + \alpha - \sigma)} + \frac{1}{\mathfrak{R}_b \sin(\vartheta + \alpha - \sigma)} \left[ \frac{1}{2} \frac{d\Theta}{d\sigma} + \frac{2(\kappa - \cos 2\alpha)}{(\kappa + 1)^2} \left( 1 - \frac{\kappa - 1}{\kappa + 1} V^2 \cos^2 \sigma \right) \operatorname{ctg} \sigma \operatorname{ctg} \alpha \right], \quad (6)$$

where  $\mathfrak{R}_b$  is the radius of curvature of the shock-wave line  $CE$  at the point  $C$ .

If the velocity of the gas that has passed through a shock wave of finite intensity remains supersonic, then on the shock wave, in particular at the point  $C$ , the known inequalities are satisfied

$$0 < \alpha < \frac{\pi}{2}, \quad \sigma < \vartheta + \alpha, \quad \sigma < \frac{\pi}{2}, \quad \frac{d\Theta}{d\sigma} > 0.$$

From these inequalities it follows that the expression multiplying  $1/\mathfrak{R}_b$  in formula (6) is positive, and a decrease in  $d\vartheta/ds$  leads to an increase in  $\mathfrak{R}_b$ . Hence, from Property 1 there in turn follows:

**Property 2.** In axisymmetric supersonic flow past a body of revolution, an increase in the radius of curvature  $\mathfrak{R}$  of the generatrix  $AB$  of the body of revolution at the point  $A$  leads to an increase in the radius of curvature  $\mathfrak{R}_b$  of the shock-wave line  $CE$  at the point  $C$ , if the points  $A$  and  $C$  are connected by a characteristic of the first family.

Property 2 permits one to conclude that, when the curvature of  $AB$  at the point  $A$  is equal to  $-\infty$ , the curvature of the shock-wave line  $CE$  at the point  $C$  is minimal.

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## CITED LITERATURE

1. A. A. Dorodnitsyn, *Collection of Theoretical Works on Aerodynamics*, Moscow, 1957, p. 77.

*Note: Figure translations are in progress. See original paper for figures.*

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