# CRITERIA FOR BOUNDEDNESS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY OF THE ARGUMENT

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## Abstract

## Full Text

## MATHEMATICS

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# CRITERIA FOR BOUNDEDNESS OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY OF THE ARGUMENT

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In our note ($^1$), criteria were established for the stability of solutions of differential equations of the form

$$\frac{dy}{dt} - A(t)y(t-a) = x(t) \qquad (0 \le t < \infty;\ a > 0).$$

In the present article we give necessary and sufficient criteria for boundedness of solutions on the half-axis $(0, \infty)$ of differential equations of the form:

$$\frac{dy}{dt} - A(t)y(t-\alpha(t)) = x(t) \qquad (\alpha(t) \ge 0);$$

$$\frac{dy}{dt} - A(t)y(t-a) - B(t)y(t) = x(t) \qquad (a > 0).$$

Consider the space $\widetilde{E}$ of continuous functions $x = x(t)$ $(-\infty < t < \infty)$ with range belonging to the complex Banach space $\tilde{\mathfrak{C}}$. Let $\widetilde{E}_0$ be the subspace of continuous functions $\{x(t)\}$ that vanish for $t < 0$. Using some results of the works ($^1$-$^4$), one can show that in the space $\widetilde{E}$ the following theorems hold.

**Theorem 1.** *Consider the boundary-value problem:*

$$\begin{aligned} \frac{dy}{dt} - \lambda y(t-\alpha(t)) &= x(t) \qquad (0 \le t < \infty), \\ y(t) &= \varphi(t) \qquad (t \le 0;\ \alpha(t) \ge 0). \end{aligned} \tag{1}$$

*Suppose that the continuous, bounded function $\alpha(t)$ admits a representation $\alpha(t) = \alpha_1(t) + \alpha_2(t)$, which satisfies the conditions:*

1) *the derivative $\alpha_1'(t)$ exists;*

2) $\lim \alpha_1'(t) = 0$ *as* $t \to +\infty$;

3) $\lim \dfrac{\alpha_2(t)}{t} = 0$ *as* $t \to +\infty$.

*Let* $\lim \alpha(t) = a > 0$ *as* $t \to +\infty$.
*In order that the boundary-value problem (1) have a bounded solution* $y(t)$ *for all bounded* $x(t)$ *and* $\varphi(t)$, *it is necessary and sufficient that all roots* $z$ *of the equation* $1 - ze^{\lambda az} = 0$ *lie outside the unit circle.*

*If, however,* $a = 0$, *then for boundedness of* $y(t)$ *it is necessary and sufficient that* $\lambda$ *lie in the open left half-plane.*

**Theorem 2.** *Consider the boundary-value problem*

$$\begin{aligned} \frac{dy}{dt} - Ay(t - \alpha(t)) = x(t) \qquad & (0 \le t < \infty), \\ y(t) = \varphi(t) \qquad & (t \le 0;\ \alpha(t) \ge 0). \end{aligned} \tag{2}$$

Let $A$ be a linear bounded operator acting in $\mathfrak{E}$; let $\alpha(t)$ satisfy the same conditions as in boundary-value problem (1).

If $a > 0$, then in order that boundary-value problem (2) have a bounded solution $y(t)$ for all bounded $x(t)$ and $\varphi(t)$, it is necessary and sufficient that, for every $\lambda$ in the spectrum of the operator $A$, all roots $z$ of the equation $1 - ze^{\lambda az} = 0$ lie outside the unit circle.

If, however, $a = 0$, then for boundedness of $y(t)$ it is necessary and sufficient that the spectrum of the operator $A$ lie in the open left half-plane.

**Theorem 3.** Consider the boundary-value problem

$$\frac{dy}{dt} - A(t)y(t - \alpha(t)) = x(t) \qquad (0 \le t < \infty),$$

$$y(t) = \varphi(t) \qquad (t \le 0;\ \alpha(t) \ge 0). \tag{3}$$

Let $A(t)$ be an operator-valued function admitting a representation $A(t) = A_1(t) + A_2(t)$, which satisfies the conditions:

1) for each fixed $t$, the operators $A_1(t)$ and $A_2(t)$ are linear, bounded, and act in $\mathfrak{E}$;
2) the family of operators $\{A_1(t)\}$ is compact: from every sequence $\{A_1(t_n)\}$ one can extract a part convergent in norm;
3) there exists a strong derivative $A_1'(t)$;
4) $\lim \|A_1'(t)\| = 0$ as $t \to +\infty$;
5) $\lim \|A_2(t)\| = 0$ as $t \to +\infty$.

Let $\alpha(t)$ satisfy the conditions of Theorem 1.

Consider the family $\{a_\omega\}$ of all possible limiting values of the function $\alpha(t)$ as $t \to +\infty$. To each $a_\omega$, evidently, there corresponds some law according to which $t$ tends to $+\infty$. Assign to each $a_\omega$ the family $\{A_\omega\}$ of those limiting operators which are generated by the family $\{A(t)\}$ under the same law of tendency of $t$ to $+\infty$.

In order that boundary-value problem (3) have a bounded solution $y(t)$ for all bounded $x(t)$ and $\varphi(t)$, it is necessary and sufficient that, for every $\lambda$ from the spectrum of at least one limiting operator $A_\omega$ corresponding to $a_\omega > 0$, all roots $z$ of the equation $1 - ze^{\lambda a_\omega z} = 0$ lie outside the unit circle.

If, however, $a_\omega = 0$, then all $\lambda$ must lie in the open left half-plane. In particular, when $a = 0$, for boundedness of the solution $y(t)$ it is necessary and sufficient that all points of the spectra of all limiting operators of the family $\{A(t)\}$ lie in the open left half-plane.

**Theorem 4.** Consider the boundary-value problem:

$$\frac{dy}{dt} - \alpha(t)y(t-a) - \beta(t)y(t) = x(t) \qquad (0 \le t < \infty),$$

$$y(t) = \varphi(t) \qquad (t \le 0;\ a > 0). \tag{4}$$

Let $\alpha(t)$ and $\beta(t)$ be complex-valued functions admitting representations $\alpha(t) = \alpha_1(t) + \alpha_2(t)$, $\beta(t) = \beta_1(t) + \beta_2(t)$, which satisfy the conditions:

1) $\alpha_1(t), \alpha_2(t), \beta_1(t), \beta_2(t)$ are continuous and bounded;
2) there exist continuous derivatives $\alpha_1'(t)$ and $\beta_1'(t)$;
3) $\lim\limits_{t\to+\infty} |\alpha_1'(t)| = 0, \quad \lim\limits_{t\to+\infty} |\beta_1'(t)| = 0;$
4) $\lim\limits_{t\to+\infty} \alpha_2(t) = 0, \quad \lim\limits_{t\to+\infty} \beta_2(t) = 0.$

Let $\{\alpha_\omega\}$ and $\{\beta_\omega\}$ be all possible limiting values of the functions $\alpha(t)$ and $\beta(t)$, generated by an arbitrary sequence $t_n \to +\infty$:

$$\alpha_\omega = \lim_{t_n\to+\infty} \alpha(t_n),$$

$$\beta_\omega = \lim_{t_n\to+\infty} \beta(t_n).$$

Then, in order that the boundary-value problem (4) have a bounded solution $y(t)$ for all bounded $x(t)$ and $\varphi(t)$, it is necessary and sufficient that all roots $z$ of the equation

$$1 - ze^{(\alpha_\omega z + \beta_\omega)a} = 0 \tag{5}$$

lie outside the unit circle.

Consider a boundary-value problem of the form

$$\begin{gathered} \frac{dy}{dt} - A(t)y(t-a) - B(t)y(t) = x(t) \qquad (0 \leqslant t < \infty), \\ y(t) = \varphi(t) \qquad (t \leqslant 0;\ a > 0), \end{gathered} \tag{6}$$

where $A(t)$, $B(t)$ are compact operator-functions, commuting with each other, acting in a Banach space $\mathfrak{S}$.

One can formulate a sufficient criterion for boundedness of solutions of the boundary-value problem (6), completely corresponding to Theorem 4; $\alpha_\omega$ and $\beta_\omega$ in the characteristic equation (5) should be replaced by points of the spectra $\lambda_\omega$ and $\mu_\omega$ of the limiting operators $A_\omega$ and $B_\omega$, generated by one and the same sequence $t_n \to +\infty$.

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