Soviet-era science, translated into English

MATHEMATICS

1958

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-195801.61526

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. G. VITUSHKIN

ON BEST APPROXIMATIONS OF DIFFERENTIABLE AND ANALYTIC FUNCTIONS

(Presented by Academician A. N. Kolmogorov on 19 XI 1957)

Let E_r^z be the space of r complex variables $z_1, z_2, \dots, z_r$; g a certain subset of the space E_r^z; F a certain collection of complex functions defined on the set g (as the distance between functions f_1 and f_2 from F one takes the supremum of $|f_1 - f_2|$ over all points of the set g).

Fix natural numbers s, n, m and two sets of arbitrary complex functions $\{f_{\alpha_1,\dots,\alpha_n}\}$ and $\{\varphi_{\beta_1,\dots,\beta_m}\}$, where $\{\alpha_i\}$ and $\{\beta_j\}$ are natural numbers such that

$$\sum_{i=1}^{n} \alpha_i \le s \quad \text{and} \quad \sum_{j=1}^{m} \beta_j \le s.$$

Denote by $e_{s,n,m}^{f}(F)$ $(f \in F)$ the lower bound of the norm of the difference

$$f - \frac{\sum\limits_{\alpha_1}^{\alpha_1+\alpha_2+\cdots+\alpha_n \le s} \sum\limits_{\alpha_2} \cdots \sum\limits_{\alpha_n} \prod\limits_{i=1}^{n} (a_i)^{\alpha_i} f_{\alpha_1,\dots,\alpha_n}(x)}{\sum\limits_{\beta_1}^{\beta_1+\beta_2+\cdots+\beta_m \le s} \sum\limits_{\beta_2} \cdots \sum\limits_{\beta_m} \prod\limits_{j=1}^{m} (b_j)^{\beta_j} \varphi_{\beta_1,\dots,\beta_m}(x)}$$

over all possible sets of complex numbers $a_1, a_2, \dots, a_n$; $b_1, b_2, \dots, b_m$, and by $e_{s,n,m}(F)$ the supremum of the quantity $e_{s,n,m}^{f}(F)$ over all possible $f \in F$.

In this note estimates (from below) are given for the quantity $e_{s,n,m}(F)$ for spaces of differentiable and analytic functions.

Theorem 1. Let $g = I_r$ be an r-dimensional closed parallelepiped; $F = F_{p+\alpha}$ the space of all $p \ge 0$ times differentiable real functions bounded on I_r by a constant $c > 0$, whose partial derivatives of order p on I_r satisfy a Hölder condition with constant $l > 0$ and exponent $\alpha > 0$.

Then for all natural s, n, m the inequality

$$e_{s,n,m}(F_{p+\alpha}) \ge \left[\frac{A(F_{p+\alpha})}{(n+m+1)\lg_2(2+s)} \right]^{(p+\alpha)/r},$$

holds, where $A(F_{p+\alpha}) > 0$ is a constant depending only on the parameters p, α, l, c, r, and I_r.

Theorem 2. Let g be a certain bounded domain of the space E_r^z; γ a bounded domain in E_r^z containing g, whose boundary is at a positive distance from g; $F = F_\gamma^g$ the space of all functions complex-analytic in g and having in the domain γ

analytic continuations bounded by the constant c (the max-norm of the modulus of the function on the closure of the domain g).

Then for all s, n, m the inequality holds:

$$e_{s,n,m}(F_\gamma^g) \geq A q^{\sqrt[r]{(n+m)\log_s(s+1)}},$$

where $A > 0$, $q > 0$ $(q < 1)$ are constants depending only on c and on the domains g and γ.

These theorems are simple consequences of the main estimates of paper (1). The meaning of Theorem 1 is that, for differentiable functions, there are scarcely any methods of approximating them by standard functions that would prove more "economical" than the approximate representation of such functions by means of algebraic polynomials.

The meaning of Theorem 2 is analogous; however, in the present case the question of the sharpness of the estimate obtained remains open, since for the known methods of approximation of analytic functions the corresponding quantity $e_{s,n,m}(F_\gamma^g)$ turns out to be considerably larger than $A[q^{\log_2(1+s)}]^{(m+n)}$.

Let us consider a concrete example. Fix $r = 1$. As g take the disk in E_1^2 $(|z| \leq 1)$, and as γ the disk $(|z| < \rho,\ \rho > 1)$. Slightly improving the methods of paper (1), one can show that, for fixed g, γ and $F = F_\gamma^g$ (see Theorem 2), the following theorem holds.

Theorem 3. If every function of the family F_γ^g on the disk $|z| \leq 1$ can be approximated with accuracy up to ε by a function of the form

$$\frac{\sum_{\alpha_1} \sum_{\alpha_2} \cdots \sum_{\alpha_n}^{\alpha_1+\alpha_2+\cdots+\alpha_n \leq s} \prod_{i=1}^{n} (a_i)^{\alpha_i} f_{\alpha_1,\ldots,\alpha_n}(x)}{\sum_{\beta_1} \sum_{\beta_2} \cdots \sum_{\beta_m}^{\beta_1+\beta_2+\cdots+\beta_m \leq s} \prod_{j=1}^{m} (b_j)^{\beta_j} \varphi_{\beta_1,\ldots,\beta_m}(x)},$$

then the numbers s, n, m, ε must satisfy the inequality

$$2(n+m)\log_2 \frac{s+1}{\varepsilon} \geq H_\varepsilon(F_\gamma^g) + o[H_\varepsilon(F_\gamma^g)] =$$

$$= \frac{1}{\log_2 \rho} \left(\log_2 \frac{c}{\varepsilon}\right)^2 + o\left[\left(\lg \frac{1}{\varepsilon}\right)^2\right],$$

where

$$H_\varepsilon(F_\gamma^g) = \frac{1}{\log_2 \rho} \left(\log_2 \frac{c}{\varepsilon}\right)^2 + o\left[\left(\lg \frac{1}{\varepsilon}\right)^2\right]$$

is the ε-entropy of the space F_γ^g (2).

Consequence of Theorem 3. For every $c > 0$, $\rho > 1$ one can specify a constant $B > 0$ such that, for all sufficiently large n, m, there exists a function $f(z)$, analytic in the disk $|z| \le \rho$ and bounded in modulus in this disk by the constant c, which cannot be approximated on the disk $|z| \le 1$ with accuracy up to

$$B\left(\frac{1}{\rho^2}\right)^{n+m}$$

by any rational function of the form

$$\sum_{k=0}^{n} a_k z^k \Big/ \sum_{k=0}^{m} b_k z^k, \qquad (1)$$

($a_0, a_1, \ldots, a_n$; $b_0, b_1, \ldots, b_m$ are complex numbers).

On the other hand, it remains unknown whether every function analytic in the disk $|z| \le \rho$ can be approximated on the disk $|z| \le 1$, with accuracy up to

$$B'\left(\frac{1}{\rho^2}\right)^{n+m}$$

(B' is some sufficiently large constant), by some rational function of the form (1). It is known only that the indicated analytic functions can be approximated by polynomials of degree n with accuracy up to

$$B'\left(\frac{1}{\rho}\right)^{n} = B'\left(\frac{1}{\rho}\right)^{n+m} \qquad (m = 0).$$

The example considered suggests that analytic functions are approximated by rational functions of the form (1) considerably more accurately than by polynomials of degree $n + m$.

Received
19 XI 1957

CITED LITERATURE

[1] A. G. Vitushkin, DAN, **114**, No. 5 (1957). [2] A. G. Vitushkin, DAN, **117**, No. 5 (1957).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.