V. I. PROTASOV

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. I. PROTASOV

ON LINEAR PARTIAL DIFFERENTIAL EQUATIONS OF INFINITE ORDER WITH CONSTANT COEFFICIENTS

(Presented by Academician S. L. Sobolev on 31 III 1958)

1. Let us consider the infinite system of differential equations of order m with constant coefficients

$$y_k^{(m)}(x) = \sum_{\nu=0}^{m-1} \sum_{n=0}^{\infty} a_n^{(\nu)} y_{n+k}^{(\nu)}(x) \tag{1}$$

with prescribed initial conditions

$$y_n^{(\nu)}(0) = c_n^{(\nu)}, \qquad n = 0, 1, 2, \dots, \qquad \nu = 0, 1, 2, \dots, m-1. \tag{2}$$

If we introduce the notation

$$A_\nu = \begin{pmatrix} a_0^{(\nu)} & a_1^{(\nu)} & a_2^{(\nu)} & \cdots \\ & a_0^{(\nu)} & a_1^{(\nu)} & \cdots \\ & & a_0^{(\nu)} & \cdots \\ & & & \cdots \end{pmatrix},$$

$$y^{(\nu)}(x) = (y_0^{(\nu)}(x),\, y_1^{(\nu)}(x),\, y_2^{(\nu)}(x),\, \dots,\, y_n^{(\nu)}(x),\, \dots), \tag{3}$$

$$c^{(\nu)} = (c_0^{(\nu)},\, c_1^{(\nu)},\, c_2^{(\nu)},\, \dots,\, c_n^{(\nu)},\, \dots), \qquad \nu = 0, 1, 2, \dots, m-1,$$

then the system (1), (2) can be written briefly in the form

$$\begin{gathered} y^{(m)} = A_{m-1} y^{(m-1)} + A_{m-2} y^{(m-2)} + \dots + A_0 y, \\ y^{(\nu)}(0) = c^{(\nu)}, \qquad \nu = 0, 1, 2, \dots, m-1. \end{gathered} \tag{4}$$

The system (4) can be reduced to a system of differential equations of the first order.

Let us introduce for consideration elements of a linear space of dimension m

$$\overline{y} = (y^{(m-1)},\, y^{(m-2)},\, \dots,\, y), \qquad \overline{c} = (c^{(m-1)},\, c^{(m-2)},\, \dots,\, c),$$

where by the generalized coordinates $y^{(\nu)}$ and $c^{(\nu)}$ $(\nu = 0, 1, 2, \dots, m-1)$ are meant points of the coordinate space in expression (3). By the derivative with respect to x of the point $\overline{y}$ is meant the point

$$\overline{y}' = (y^{(m)},\, y^{(m-1)},\, \dots,\, y').$$

We form from the matrices A_ν $(\nu = 0, 1, 2, \dots, m-1)$ the matrix

$$B = \begin{pmatrix} A_{m-1} & A_{m-2} & \cdots & A_0 & \\ E & 0 & \cdots & 0 & \\ 0 & E & \cdots & 0 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & E & 0 \end{pmatrix}, \qquad E \text{ is the identity matrix.}$$

With the aid of the matrix B, system (4) can be written in the form

$$\bar{y}' = B\bar{y}, \qquad \bar{y}(0) = \bar{c}. \tag{5}$$

Formally, the solution of system (5) can be written as [1]:

$$\bar{y} = e^{xB}\bar{c}, \qquad \text{where} \qquad e^{xB} = E + xB + \frac{x^2}{2!}B^2 + \cdots + \frac{x^n}{n!}B^n + \cdots \tag{6}$$

Expression (6) is indeed a solution of system (5) when the series entering expression (6) converge absolutely. If the coordinates of the points $h(h_0, h_1, h_2, \dots, h_n, \dots)$ and $f(f_0, f_1, f_2, \dots, f_n, \dots)$ for all n satisfy the inequalities $|h_n| \leq |f_n|$ (or $|h_n| < |f_n|$), then this circumstance will be written as follows: $|h| \leq |f|$ $(|h| < |f|)$.

Let us denote

$$\left|y^{(m-1)}(x)\right| = \left(\left|y_1^{(m-1)}(x)\right|, \left|y_1^{(m-1)}(x)\right|, \left|y_2^{(m-1)}(x)\right|, \dots\right),$$

$$\sum_{\nu=0}^{m-1} \left|c^{(\nu)}\right| = \left(\sum_{\nu=0}^{m-1} \left|c_0^{(\nu)}\right|, \sum_{\nu=0}^{m-1} \left|c_1^{(\nu)}\right|, \sum_{\nu=0}^{m-1} \left|c_2^{(\nu)}\right|, \dots\right),$$

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ & a_0 & a_1 & \cdots \\ & & a_0 & \cdots \\ & & & \ddots \end{pmatrix}, \qquad \text{where} \quad a_n = \max_{0 \le \nu \le m-1} \left\{ |a_n^{(\nu)}| \right\}. \tag{7}$$

In the adopted notation, the $(m-1)$-st derivative of the solution of system (4) can, for $|x| \le r$, be estimated by the inequality

$$\left| y^{(m-1)} \right| \le e^{r(A+E)} \sum_{\nu=0}^{m-1} \left| c^{(\nu)} \right|. \tag{8}$$

With the aid of inequality (8) and the results of the article [2], one can prove the following theorems. The first four theorems are conveniently formulated with the aid of the function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The coefficients a_n are the elements of the matrix A in (7).

Theorem 1. If the function $F(z)$ is a polynomial of degree s with leading coefficient a_s, then, under initial conditions $c_k^{(\nu)} = y_k^{(\nu)}(0)$ such that

$$(R|a_s|es)^{1/s} \lim_{n\to\infty} \frac{\sqrt[n]{\left|c_n^{(\nu)}\right|}}{n^{1/s}} \le 1 \qquad (\nu = 0, 1, 2, \dots, m-1),$$

the system of equations (1) has a unique solution, analytic in the disk $|x| < R$.

Theorem 2. If $F(z)$ is a polynomial of degree s, and for the initial conditions the relation

$$\lim_{n\to\infty} \frac{\sqrt[n]{\left|c_n^{(\nu)}\right|}}{n^{(1-\varepsilon)/s}} < \infty \qquad (\nu = 0, 1, 2, \dots, m-1), \quad \varepsilon > 0,$$

is satisfied, then the solution of system (1) is analytic on the whole plane and is unique.

Theorem 3. If, for all sufficiently large $|z|$, the inequality

$$|F(z)| < e^{\sigma |z|^{\rho}},$$

then, under initial conditions satisfying the relation

$$\left(e^{1/\sigma}\right)^{1/\rho} \lim_{n\to\infty} \frac{\sqrt[n]{\left|c_n^{(\nu)}\right|}}{(\ln n)^{1/\rho}} < 1 \qquad (\nu = 0, 1, 2, \dots, m-1),$$

the solution of system (1) is analytic in the entire plane and is unique.

Theorem 4. If the function $F(z)$ is analytic in a circle of radius R, then, under initial conditions satisfying the inequality

$$\lim_{n\to\infty} \frac{\sqrt[n]{\left|c_n^{(\nu)}\right|}}{R} < 1 \qquad (\nu = 0, 1, 2, \dots, m-1),$$

the solution of system (1) exists among entire functions and is unique.

Theorem 5. If $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$, then, under the initial conditions

$$y_n^{(\nu)}(0) = c_n^{(\nu)}, \qquad \lim_{n\to\infty} 2\sqrt[n]{b_n \left|c_n^{(\nu)}\right|} < 1, \qquad \nu = 0, 1, 2, \dots, m-1,$$

where $b_0 = a_0,\ b_n = \max\{a_0^n, a_1^n, a_2^n, \dots, a_k^{n/k}, \dots, a_n\}$, the system of equations (1) has a unique solution, analytic in the entire plane.

2. Using the results on solutions of the system of linear equations (1) under conditions (2), one can obtain theorems on solutions of an equation in partial derivatives.

It is required to find a function $u(x, z)$ satisfying the equation

$$\frac{\partial^m u}{\partial x^m} = \sum_{\nu=0}^{m-1} \sum_{n=0}^{\infty} a_n^{(\nu)} \frac{\partial^{\nu+n} u}{\partial x^\nu \partial z^n}, \tag{9}$$

if the functions are given

$$\left.\frac{\partial^\nu u}{\partial x^\nu}\right|_{x=0} = v_\nu(z), \qquad \nu = 0, 1, 2, \dots, m-1. \tag{10}$$

We shall seek the solution of (9) under conditions (10), among functions analytic in some domain, in the form of an absolutely convergent series

$$u(x, z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} y_n(x), \tag{11}$$

where the functions $y_n(x)$ $(n = 0, 1, 2, \dots)$ are to be determined. Substituting expression (11) into equation (9) and conditions (10), and assuming that the series obtained thereby are absolutely convergent, we arrive at the fact that the functions $\{y_n(x)\}$ must satisfy system (1) under the initial conditions

$$y_n^{(\nu)}(0) = c_n^{(\nu)} = v_\nu^{(n)}(0), \qquad n = 0, 1, 2, \dots, \quad \nu = 0, 1, 2, \dots, m-1. \tag{12}$$

With the aid of the stated results on solutions of the system (1), (2), one can obtain, for an equation of the form

$$\frac{\partial^m u}{\partial x^m} = \sum_{\nu=0}^{m-1} \sum_{n=0}^{s} a_n^{(\nu)} \frac{\partial^{n+\nu} u}{\partial x^\nu \partial z^n} \tag{13}$$

under the conditions

$$v_\nu(z) = \left.\frac{\partial^\nu u}{\partial x^\nu}\right|_{x=0}, \qquad \nu = 0, 1, 2, \dots, m-1$$

the following theorems.

Theorem 6. If the functions $v_\nu(z)$ $(\nu = 0, 1, 2, \dots, m-1)$ are entire of order $\frac{s}{s-1}$ $(s > 1)$ and of type σ, then the solution $u(x, z)$ of equation

$$\tag{13}$$

is analytic in x in the disk $|x| < R$, where

$$R < \frac{\left(\frac{s-1}{s}\right)^{s-1}}{|a_s|^{1/s}}$$

and

$$|a_s| = \max_{0 \le \nu \le m-1} \{|a_s^{(\nu)}|\},$$

and, for any fixed x in the disk $|x| < R$, has growth in z not exceeding order $\frac{s}{s-1}$ and of normal type.

Theorem 7. If the functions $v_\nu(z)$ $(\nu = 0, 1, 2, \dots, m-1)$ are entire functions of order of growth $\frac{s}{s-\theta}$, where $0 < \theta < 1$, of normal type, then equation (13) has a unique solution $u(x, z)$, analytic in the variables x and z in the whole plane. For fixed x, the function $u(x, z)$ grows in z not faster than order $\frac{s}{s-\theta}$ and is of normal type.

For equation (9) of general form, denote by $f(\zeta)$ the function

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n,$$

where

$$a_n = \max_{0 \le \nu \le m-1} \{|a_n^{(\nu)}|\}.$$

Theorem 8. If, for all sufficiently large $|\zeta|$,

$$|f(\zeta)| < e^{\sigma |\zeta|^\rho}$$

($\sigma > 0, \rho > 0$), and the functions $v_\nu(z)$ ($\nu = 0, 1, 2, \dots, m-1$) are entire functions of growth not exceeding first order and of normal type, then equation (9) has a unique analytic solution, entire in x and z. For one fixed variable, the function $u(x, z)$ grows with respect to the other variable no faster than a function of first order and of normal type.

Theorem 9. If the function

$$f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$$

is analytic in the disk $|\zeta| < R$, and the functions

$$v_\nu(z) = \frac{\partial^\nu u}{\partial x^\nu}\bigg|_{x=0} \qquad (\nu = 0, 1, 2, \dots, m-1)$$

are entire functions whose order and type are less than R, then the solution of equation (9) is entire in the variables x, z and is unique among analytic functions. For fixed x, the solution $u(x, z)$ has, in z, the growth of an entire function not exceeding first order and of type less than R, and for fixed z the function $u(x, z)$ with respect to x is of order not exceeding the first and of normal type.

When

$$\lim_{n \to \infty} \sqrt[n]{a_n} = \infty,$$

where

$$a_n = \max_{0 \le \nu \le m-1} \{|a_n^{(\nu)}|\},$$

the following theorem holds:

Theorem 10. If

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \infty,$$

then under conditions (10)

$$\left.\frac{\partial^\nu u}{\partial x^\nu}\right|_{x=0} = v_\nu(z) = \sum_{n=0}^{\infty} \frac{c_n^{(\nu)}}{n!} z^n,$$

whose growth is determined by the inequality

$$\lim_{n\to\infty} 2\sqrt[n]{b_n |c_n^{(\nu)}|} < 1,$$

where

$$b_0 = a_0, \qquad b_n = \max\{a_0^n, a_1^n, \dots, a_k^{n/k}, \dots, a_n\},$$

equation (9) has a unique solution, analytic in the variables x, z in the whole plane. For fixed z, the solution has in x growth not exceeding first order and of normal type, and for fixed x it has growth not exceeding first order and of minimal type.

Taganrog Radio-Engineering Institute

Received
23 I 1958

References

1. F. R. Gantmakher, *Matrix Theory*, 1953.

2. V. I. Protasov, DAN, **105**, No. 2 (1955).

Note: Figure translations are in progress. See original paper for figures.