# MATHEMATICS

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1958

SovietRxiv

## Abstract

## Full Text

# MATHEMATICS

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## ON THE BEHAVIOR OF SOLUTIONS OF THE CAUCHY PROBLEM FOR SOME QUASILINEAR EQUATIONS AS TIME INCREASES WITHOUT BOUND

*(Presented by Academician I. G. Petrovskii, 10 I 1958)*

The paper investigates the behavior of solutions of the Cauchy problem for a quasilinear parabolic equation with two independent variables, and also the behavior of generalized solutions of the Cauchy problem for a quasilinear first-order equation as $t \to \infty$.* We shall restrict ourselves to considering equations of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial x^2}, \qquad \varepsilon > 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(u)}{\partial x} = 0, \tag{2}$$

which are model equations for gas dynamics equations. The question of the behavior of the solution of the Cauchy problem as $t \to \infty$ for an equation of the form $\partial u/\partial t = \varepsilon\, \partial^2 u/\partial x^2 + F(u)$ was considered in ([1]). For equations (1) and (2), with $\varphi(u) = u^2/2$ and with an initial function summable on the whole axis, this question was studied in ([2]).

We shall consider solutions of the Cauchy problem for equations (1) and (2) in the half-plane $t \geq 0$ with initial condition

$$u\big|_{t=0} = u_0(x), \qquad -\infty < x < +\infty, \tag{3}$$

where $u_0(x)$ is a bounded measurable function. The proof of existence, uniqueness, and properties of these solutions is given in ([3]).

1. We shall assume that $\varphi(u)$ has continuous derivatives up to the fourth order, $\varphi'' \geq \mu > 0$, and that $u_0(x) \to u_-$ as $x \to -\infty$ and $u_0(x) \to u_+$ as $x \to +\infty$. It has been proved that there exists in $R\{t \geq 0, -\infty < x < +\infty\}$ a bounded function $u_\varepsilon(t,x)$, satisfying equation (1) for $t > 0$ and assuming the initial values (3) in the weak sense (see ([3]), p. 35).

**Theorem 1.** *Let $u_- > u_+$, and suppose that for $u_0(x)$ the integrals*

$$\int_{-\infty}^{0} (u_0(x) - u_-)\, dx, \qquad \int_{0}^{+\infty} (u_0(x) - u_+)\, dx \tag{4'}$$

*exist and their sum is equal to A. Equation (1) has a unique solution $\tilde{u}_\varepsilon(x-Kt)$, depending only on $x-Kt$, where $K = [\varphi(u_+)-\varphi(u_-)]/[u_+-u_-]$, and such that*

$$\int_{-\infty}^{0} (\tilde{u}_\varepsilon(x) - u_-)\, dx + \int_{0}^{+\infty} (\tilde{u}_\varepsilon(x) - u_+)\, dx = A.$$

* This problem was brought to our attention by I. M. Gel' fand.

As $t \to \infty$

$$|\tilde{u}_\varepsilon(x - Kt) - u_\varepsilon(t, x)| \to 0$$

uniformly with respect to $x$. If, for $u_0(x)$, the additional conditions

$$\left|\int_{-\infty}^{x} (u_0(x) - u_-)\, dx\right| \le M_1 e^{\alpha_1 x}, \qquad \left|\int_{x}^{+\infty} (u_0(x) - u_+)\, dx\right| \le M_1 e^{-\alpha_1 x} \tag{4}$$

are satisfied for some constants $\alpha_1 > 0$ and $M_1 > 0$, then

$$|\tilde{u}_\varepsilon(x - Kt) - u_\varepsilon(t, x)| \le M_2 e^{-\beta t}$$

for all $x$ and $t$, where $\beta > 0$, $M_2 > 0$ are certain constants.

First we shall prove Theorem 1 for a twice continuously differentiable function $u_0(x)$, under the condition that $u_0'(x) \to 0$ as $x \to \pm\infty$ and $u_0''(x)$ is bounded. In this case, as shown in (4), $u_\varepsilon(t,x)$ is continuous and bounded in $R$, together with all derivatives entering equation (1), and for $t > 0$ this equation may be differentiated with respect to $x$. It suffices to prove Theorem 1 for $K = 0$. Indeed, make the change of variables $x - Kt = x_1,\ t = t_1$. In the new variables equation (1) is written in the form

$$\partial u/\partial t_1 + \partial \varphi_1(u)/\partial x_1 = \varepsilon\, \partial^2 u/\partial x_1^2,$$

where $\varphi_1(u) = \varphi(u) - Ku$, and consequently $\varphi_1(u_+) = \varphi_1(u_-)$ and $K_1 = 0$. The proof of Theorem 1 under these assumptions is based on the following lemmas.

**Lemma 1.** Let $u(t,x)$ be a continuous function in $R$, having for $t > 0$ the derivatives $u_x, u_t, u_{xx}$, and suppose that when $u(t,x) \le 0$ the condition

$$\varepsilon\, \partial^2 u/\partial x^2 - \partial u/\partial t + a(t,x,u)\, \partial u/\partial x + c(t,x,u)u \le 0$$

is satisfied, where $a$ and $c$ are bounded functions and $c \le 0$ for $u \le 0$. If $u(0,x) \ge 0$ and $u(t,x) \ge M(t)\sqrt{x^2+1}$, where $M(t)$ is a continuous function, then $u(t,x) \ge 0$ in $R$.

With the aid of this lemma, Lemmas 2 and 3 are established.

**Lemma 2.** As $x \to \pm\infty$, the solution $u_\varepsilon(t,x)$ of problem (1), (3) tends respectively to $u_+$ and $u_-$, uniformly on each finite interval of variation of $t$.

**Lemma 3.** As $x \to \pm\infty$, $\partial u_\varepsilon/\partial x \to 0$ uniformly on each finite interval of variation of $t$.

**Lemma 4.** For any $t \geq 0$ and for any $x$, the integrals

$$\int_{-\infty}^{x} (u_\varepsilon(t,x) - u_-)\,dx, \qquad \int_{x}^{+\infty} (u_\varepsilon(t,x) - u_+)\,dx$$

exist.

The proof of this assertion, for example for the first integral, is based on the fact that the function $v_\varepsilon(t,x)$, which satisfies in $R$ the equation

$$\partial v/\partial t + \varphi(u) - \varphi(u_+) = \varepsilon\,\partial^2 v/\partial x^2$$

and the initial condition

$$v_\varepsilon(0,x) = \int_{-\infty}^{x} (u_0(x) - u_-)\,dx,$$

for any $t \geq 0$ tends to zero as $x \to -\infty$, and that

$$\partial v_\varepsilon/\partial x = u_\varepsilon(t,x) - u_-.$$

**Lemma 5.** For any $t \geq 0$,

$$\int_{-\infty}^{0} (u_\varepsilon(t,x) - u_-)\,dx + \int_{0}^{+\infty} (u_\varepsilon(t,x) - u_+)\,dx = A.$$

This equality is established by differentiating its left-hand side with respect to $t$, taking into account equation (1) and Lemmas 2 and 3.

**Lemma 6.** As $x \to -\infty$, $u_\varepsilon(t,x) \to u_-$, and as $x \to +\infty$, $u_\varepsilon(t,x) \to u_+$, uniformly in $t$ and $\varepsilon$ $(t \geq 0,\ 0 < \varepsilon \leq 1)$.

To prove this lemma, it is first established that the functions

$$v_-(t,x) = \int_{-\infty}^{x} (u_\varepsilon(t,x) - u_-)\,dx \quad \text{and} \quad v_+(t,x) = \int_{x}^{+\infty} (u_\varepsilon(t,x) - u_+)\,dx$$

uniform-

but as $t$ and $\varepsilon$ tend to zero as $x \to \pm\infty$. In this case $v_-(t,x)$ is estimated from below by the function

$$w_1 = \int_{-\infty}^{x} (\tilde{u}_\varepsilon(x+c) - u_-)\,dx - \delta,$$

where $c$ is a sufficiently large constant, and from above by the function

$$w_2 = M_3 e^{\alpha_3 x} + \delta,$$

where $M_3 > 0$ and $\alpha_3 > 0$ are certain constants, $\delta > 0$ is an arbitrarily small number, since $v_- - w_1$ and $w_2 - v_-$ satisfy equations to which the maximum principle is applicable. For any $\varepsilon > 0$, $\partial u_\varepsilon / \partial x < c_1$, and therefore Lemma 6 follows from the assertion just stated for $v_+$ and $v_-$.

To prove Theorem 1 for $K = 0$, using the preceding lemmas, it is established that

$$z_\varepsilon(t,x) = \int_{-\infty}^{x} (u_\varepsilon(t,x) - \tilde{u}_\varepsilon(x))\, dx$$

satisfies an equation for which the maximum principle is valid, and for all $x$ and $t \geq 0$ the inequality

$$|z_\varepsilon(t,x)| \leq \delta + M e^{-\beta t - \alpha\theta(x)}, \tag{5}$$

holds, where $M, \beta, \alpha$ are certain positive numbers depending on $\delta$; $\delta > 0$ is an arbitrarily small number; and $\theta(x) > 0$ is a suitably chosen function. If conditions (4) are satisfied, then (5) is valid with $\delta = 0$ and for certain $M, \alpha, \beta$. Since $\partial(u_\varepsilon - \tilde{u}_\varepsilon)/\partial x < c_2$ for all $t$, the assertion of Theorem 1 follows from (5) for a smooth function $u_0(x)$ satisfying the conditions stated above. To prove Theorem 1 for an arbitrary bounded measurable function $u_0(x)$, we construct a sequence of smooth functions $u_0^n(x)$, satisfying the conditions indicated earlier, and such that

$$\int_{-\infty}^{+\infty} |u_0(x) - u_0^n(x)|\, dx \to 0$$

as $n \to \infty$. Taking into account (3), that for the corresponding solutions of the Cauchy problem

$$\int_{-\infty}^{+\infty} |u_\varepsilon(t,x) - u_\varepsilon^n(t,x)|\, dx \to 0$$

as $n \to \infty$ and for any $t$, and using the already proved Theorem 1 for the functions $u_0^n(x)$, we obtain the complete proof of Theorem 1.

As examples show, the existence of the integrals (4′) is an essential condition for the validity of Theorem 1.

**Theorem 2.** Suppose $u_+ = u_- = a$. As $t \to \infty$, $|u_\varepsilon(t,x) - a| \to 0$ uniformly for all $x$. If

$$|u_0(x) - a| \leq M_1 e^{-\alpha_1 |x|}$$

for some $\alpha_1 > 0$ and $M_1 > 0$, then for all $x$ and $t \geq 0$

$$|u_\varepsilon(t,x) - a| \leq M t^{1/2} |\ln t|^\beta, \tag{6}$$

where $M > 0$ and $\beta > 0$ are constants.

To prove Theorem 2 we use the following proposition: if $u_0(x) \geq u_0^1(x)$, then for the corresponding solutions of the Cauchy problem (1), (3) the inequality

$u_\varepsilon(t,x) \geq u_\varepsilon^1(t,x)$ holds. Let $u_0^1(x) = a + 2\delta$ for $x \leq -N$, $u_0^1(x) = a + \delta$ for $x \geq N$, and $u_0^1(x) \geq u_0(x)$ for all $x$. Then $u_\varepsilon(t,x) \leq u_\varepsilon^1(t,x)$. But, by Theorem 1, $|u_\varepsilon^1(t,x) - a| \leq 3\delta$, if $t \geq T$ and $T$ is sufficiently large. Consequently, for $t \geq T$, $u_\varepsilon(t,x) \leq a + 3\delta$. In the same way we obtain an estimate from below. Estimate (6) is established on the basis of studying the dependence of $T$ on $\delta$.

**Theorem 3.** Suppose $u_+ > u_-$, and $H(s)$ is the function defined by the equality $s = \varphi'(H(s))$ for $\varphi'(u_-) \leq s \leq \varphi'(u_+)$, $H(s) = u_-$ for $s \leq \varphi'(u_-)$, and $H(s) = u_+$ for $s \geq \varphi'(u_+)$. Then the solution $u_\varepsilon(t,x)$ of problem (1), (3) tends uniformly in $x$ and $\varepsilon$ to $H(x/t)$ as $t \to \infty$.

To prove Theorem 3 the following lemma is established:

**Lemma 7.** For any $\delta > 0$ there exists $N(\delta) > 0$ such that

$$|u_\varepsilon(t,x) - u_-| \leq \delta$$

for

$$x - \varphi'(u_-)t \leq -N,$$

and

$$|u_\varepsilon(t,x) - u_+| \leq \delta$$

for

$$x - \varphi'(u_+)t \geq N.$$

Using this lemma, the assertion of Theorem 3 for $\varphi'(u_-) \ll x/t \ll \varphi'(u_+)$ is easily obtained by considering the equation satisfied by $u_\varepsilon(t,x) - H(x/t)$.

2. Consider the generalized solution $u(t,x)$ of the Cauchy problem for $t \geqslant 0$ for equation (2) with initial condition (3). It has been proved (5) that $u_\varepsilon(t,x) \to u(t,x)$ as $\varepsilon \to 0$ at every point of continuity of $u(t,x)$, and that the set of discontinuity points of $u(t,x)$ on each straight line $t = t_0 > 0$ is at most countable (see also (3)).

**Theorem 4.** Let $u_- > u_+$,

$$\int_{-\infty}^{0} (u_0(x) - u_-)\,dx + \int_{0}^{+\infty} (u_0(x) - u_+)\,dx = A,$$

$u(t,x)$ be the solution of problem (2), (3). Let $\sigma(x) = u_-$ for $x \leqslant x_0$ and $\sigma(x) = u_+$ for $x \geqslant x_0$, where $x_0$ is determined so that

$$\int_{-\infty}^{0} (\sigma(x) - u_-)\,dx + \int_{0}^{+\infty} (\sigma(x) - u_+)\,dx = A.$$

Then, as $t \to \infty$, $|\sigma(x - Kt) - u(t,x)| \to 0$ uniformly in $x$ outside the region $x_0 - \delta \ll x - Kt \ll x_0 + \delta$, where $\delta > 0$ is an arbitrary number, $K = [\varphi(u_+) - \varphi(u_-)]/[u_+ - u_-]$.

This theorem is established with the aid of Lemmas 5, 6 and certain properties of $u(t, x)$ given in § 6 of paper (3).

**Theorem 5.** If $u_- > u_+$ and $u_0(x) = u_-$ for $x < -N$ and $u_0(x) = u_+$ for $x > N$, where $N > 0$ is some number, then $u(t, x)$ coincides with the function $\sigma(x - Kt)$, defined in Theorem 4, for $t \geqslant T$, where $T$ is some number.

**Theorem 6.** Let $u_+ = u_- = a$. Then $|u(t, x) - a| \to 0$ as $t \to \infty$ uniformly in $x$.

This theorem is proved with the aid of Theorem 5 analogously to the way Theorem 2 was proved with the aid of Theorem 1.

**Theorem 7.** Let $u_- < u_+$. Then $u(t, x)$ tends uniformly in $x$, as $t \to \infty$, to the function $H(x/t)$ defined in Theorem 3.

This theorem follows from Theorem 3 and the convergence of $u_\varepsilon(t, x)$ to $u(t, x)$ as $\varepsilon \to 0$.

Received
10 I 1958

## References

[1] A. N. Kolmogorov, I. G. Petrovskii, N. S. Piskunov, *Bull. Moscow State Univ.*, Sect. A, Math. and Mech., vol. 6 (1937).
[2] E. Hopf, *Comm. on pure and Appl. Math.*, 3, No. 3, 201 (1950).
[3] O. A. Oleinik, *Uspekhi Mat. Nauk*, 12, issue 3 (75), 3 (1957).
[4] O. A. Oleinik, T. D. Ventsel, *Mat. Sbornik*, 4 (83), No. 1, 105 (1957).
[5] O. A. Oleinik, *Uspekhi Mat. Nauk*, 9, issue 3 (61), 231 (1954).

*Note: Figure translations are in progress. See original paper for figures.*