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LOCAL STRESSES IN THE BENDING OF A CIRCULAR PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE

`![Fig. 1](#)`

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

THEORY OF ELASTICITY

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LOCAL STRESSES IN THE BENDING OF A CIRCULAR PRISMATIC BAR WITH AN ECCENTRIC ELLIPTICAL HOLE

(Presented by Academician N. I. Muskhelishvili, 17 V 1958)

1. Let us consider the problem of transverse bending of a circular prismatic bar weakened in the longitudinal direction by an elliptical cavity whose axis is located at some distance from the axis of the solid bar (Fig. 1). To determine the stressed state of the bar under transverse bending by a concentrated force, it is necessary to find the complex torsion function for the given profile and another regular function of the complex variable $z = x + iy$, satisfying certain boundary conditions. The torsion function for the problem considered here was determined in our work ([1]). We shall now determine a function regular in the domain S and satisfying the boundary conditions ([2])

Fig. 1

$$\varphi_1(t) + \overline{\varphi_1(t)} = 2F_1(t) + D_1 \qquad \text{on } L_1; \tag{1}$$

$$\varphi_1(t) + \overline{\varphi_1(t)} = 2F_2(t) \qquad \text{on } L_2; \tag{2}$$

$$F_j(t) = -(1 - {}^1/{}_2\sigma)\, {}^1/{}_3 y^3 = {}^1/{}_2 \sigma x^2 y + 2(1+\sigma)\int xy\,dx; \tag{3}$$

D_1 is a certain constant; σ is Poisson' s ratio.

2. We take the function mapping the exterior of the ellipse onto the exterior of the circle γ of radius $\rho > 1$ with center at the origin in the ζ-plane in the form

$$z - ie = A\left(\zeta - \frac{1}{\zeta}\right); \qquad \zeta = \frac{z - ie + \sqrt{(z-ie)^2 + 4A^2}}{2A} \tag{4}$$

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$$\left(A = \frac{\sqrt{a^2 - b^2}}{2}, \qquad \rho = \sqrt{\frac{a+b}{a-b}} > 1 \right).$$

The radical is positive imaginary for the affix z lying on the positive imaginary axis and having modulus greater than $e + 2A$.

From formulas (3), after simple transformations, we obtain:

$$F_1(t) = F^*(\tau) = q_0 + q_1 \left(\frac{\tau}{\rho} - \frac{\rho}{\tau} \right) + q_2 \left(\frac{\tau}{\rho^2} + \frac{\rho^2}{\tau} \right) + q_3 \left(\frac{\tau^3}{\rho^3} - \frac{\rho^3}{\tau^3} \right) \qquad \text{on } \gamma;$$

$$F_2(t) = iR^3 \left\{ e_1 \left(\frac{t}{R} - \frac{R}{t} \right) + e_3 \left[\left(\frac{t}{R} \right)^3 - \left(\frac{R}{t} \right)^3 \right] \right\} \qquad \text{on } L_2.$$

Here τ is the affix of a point of γ; t is the affix of a point of L_2;

$$e_1 = {}^3/_8 + {}^1/_4\sigma; \qquad e_3 = -{}^1/_8;$$

$$q_0 = -{}^1/_2 e \left\{ e^2 + {}^3/_2 A^2 (\rho + \rho^{-1})^2 \right\} + {}^1/_2 \sigma e \left\{ {}^1/_2 e^2 + 2A^2 \right\};$$

$$q_1 = {}^1/_2 A(\rho + \rho^{-1}) \left\{ e^2 + {}^1/_4 A^2 \left[3(\rho^2 + \rho^{-2}) - 2 \right] + {}^1/_2 \sigma \left[-e^2 + A^2(\rho^2 + \rho^{-2} - 3) \right] \right\} i;$$

$$q_2 = {}^1/_2 A^2 e(\rho^2 + \rho^{-2} - \sigma); \qquad q_3 = -{}^1/_4 A^3 (\rho + \rho^{-1}) \left\{ {}^1/_2(\rho^2 + \rho^{-2}) - {}^1/_3 - {}^1/_2\sigma \right\} i.$$

Following the method of D. I. Sherman ([3]), introduce on L_2

$$\varphi_1(t) - \overline{\varphi_1(t)} = 2\omega(t). \tag{5}$$

With the aid of (2), (5), and the property of the Cauchy-type integral, introduce in the domain S a new function, regular and analytically continuable outside the ellipse and vanishing at infinity,

$$\varphi(z) = \varphi_1(z) - \frac{1}{2\pi i} \int_{L_2} \frac{\omega(t_2)}{t_2 - z} \, dt_2 - iR^3 \left[e_1 \frac{z}{R} + e_3 \left(\frac{z}{R} \right)^3 \right]. \tag{6}$$

Taking the function (6) into account in the first boundary condition (1), passing on the basis of (4) to the variable ζ, and putting $\varphi^*(\zeta) = \varphi(z)$, we shall have

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$$\varphi^*(\zeta) = \sum_{n=1}^{\infty} \lambda_n \frac{\rho^n}{\zeta^n},$$

$$2q_0 + 2e\{e_1 R^2 - e_3(e^2 + 6A^2)\} - (\alpha_0 + \overline{\alpha_0}) + D_1 - 2b_0 = 0, \tag{7}$$

$$\lambda_n = -(1 + (-1)^n \rho^{2n}) b_n + (1 - (-1)^n \rho^{2n}) d_n - p_1 \varepsilon_n^{(1)} + p_2 \varepsilon_n^{(2)} - p_3 \varepsilon_n^{(3)}, \tag{8}$$

$$\varepsilon_n^{(\nu)} = 1 \quad \text{for } n = \nu; \qquad \varepsilon_n^{(\nu)} = 0 \quad \text{for } n \neq \nu.$$

$$p_1 = 2q_1 + \{(\rho + \rho^{-1}) A[3e_3(e^2 + A^2) - R^2 e_1]\} i; \qquad p_2 = 2q_2 + 3eA^2 e_3(\rho^2 + \rho^{-2});$$

$$p_3 = 2q_3 - A^3 e_3(\rho^3 + \rho^{-3}) i; \qquad b_0 = \sum_{k=2}^{\infty *} (-1)^{k/2} C_k^{k/2} \left(\frac{A}{R-e}\right)^k \frac{\alpha_k + \overline{\alpha_k}}{2}; \tag{9}$$

$$b_n = \frac{1}{\rho^n} \sum_{k=|n|}^{\infty *} (-1)^{(n+k)/2} C_k^{(n+k)/2} \left(\frac{A}{R-e}\right)^k \frac{\alpha_k + \overline{\alpha_k}}{2}, \qquad n = \mp 1, \mp 2, \ldots;$$

$$d_n = -\frac{1}{\rho^n} \sum_{k=n}^{\infty *} (-1)^{(n+k)/2} C_k^{(n+k)/2} \left(\frac{A}{R-e}\right)^k \frac{\alpha_k - \overline{\alpha_k}}{2}, \qquad n = \mp 1, \mp 2, \ldots$$

The functionals α_k $(k = 0, 1, 2, \ldots)$, equal to

$$\alpha_k = \frac{(R-e)^k}{2\pi i} \int_{L_2} \frac{\omega(t_2)}{(t_2 - ie)^{k+1}} \, dt_2, \tag{10}$$

are, generally speaking, complex quantities.

3. On the basis of (6), from (5) we shall have:

$$\omega(t) = \varphi(t) - \overline{\varphi(t)} + iR^3 \left[e_1 \left(\frac{t}{R} + \frac{R}{t} \right) + e_3 \left(\left(\frac{t}{R} \right)^3 + \left(\frac{R}{t} \right)^3 \right) \right] + \beta_0; \tag{11}$$

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here β_0 is a purely imaginary constant,

$$\beta_0 = \frac{1}{2\pi i} \int_{L_2} \frac{\omega(t_2)}{t_2} \, dt_2.$$

Taking (7) into account and adopting the notation

$$\beta_m = \frac{R^m}{2\pi i} \int_{L_2} \frac{\omega(t)}{t^{m+1}} \, dt, \tag{12}$$

from (11), after some reasoning, we shall have

$$\beta_m = \sum_{n=1}^{m} \rho^n \overline{\lambda_n} J_{m,n} + iR^3 \left(\varepsilon_m^{(1)} e_1 + \varepsilon_m^{(3)} e_3 \right); \tag{13}$$

$$J_{m,n} = -\left(\frac{e}{R}\right)^m \frac{A}{e} (-i)^{m-1} \times$$

$$\times \sum_{\nu_1 = n-1}^{N(m,n)*} (-1)^{(\nu_1 - n + 1)/2} i^{\nu_1} \left(\frac{A}{e}\right)^{\nu_1} C_{m-1}^{\nu_1} \left(C_{\nu_1}^{(\nu_1 - n + 1)/2} - C_{\nu_1}^{(\nu_1 - n - 1)/2} \right), \tag{14}$$

$$N(m,n) = \begin{cases} m-1, & \text{if } m \text{ and } n \text{ have the same parity;} \\ m-2, & \text{if } m \text{ and } n \text{ have different parity.} \end{cases}$$

As follows from formula (4), for m and n of the same parity $J_{m,n}$ are real quantities; for m and n of different parity $J_{m,n}$ are purely imaginary.

Substituting into (10) the expansion of a purely imaginary function continuous on L_2,

$$\omega(t) = \beta_0 + \sum_{k=1}^{\infty} \left\{ \beta_k \left(\frac{t}{R}\right)^k - \bar{\beta}_k \left(\frac{\bar{t}}{R}\right)^k \right\}, \tag{15}$$

after some transformations and reasoning we shall have

$$\alpha_k = \left(\frac{R}{e} - 1\right)^k \sum_{k_1 = k}^{\infty} (-1)^{k_1 - k} C_{-k-1}^{k_1 - k} i^{k_1 - k} \left(\frac{e}{R}\right)^{k_1} \beta_{k_1}. \tag{16}$$

On the basis of equalities (9) and (16), from (13) we finally obtain

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$$\sum_{k_1=1}^{\infty}\left(q_{k_1,m}\beta_{k_1}+q'_{k_1,m}\bar{\beta}_{k_1}\right)=f_m,$$

$$q_{m,m}=1-\sum_{n=1}^{E(m,m)}J_{m,n}\mu^*_{m,n},\qquad q_{k_1,m}=-\sum_{n=1}^{E(k_1,m)}J_{m,n}\mu^*_{k_1,n},$$

$$q'_{k_1,m}=-\sum_{n=1}^{E(k_1,m)}J_{m,n}\mu_{k_1,n}; \tag{17}$$

$$f_m=p_1\rho J_{m,1}+p_2\rho^2J_{m,2}+p_3\rho^3J_{m,3}+iR^3\left(\varepsilon_m^{(1)}e_1+\varepsilon_m^{(3)}e_3\right);$$

$$\mu_{k_1,n}=-\left(\frac{e}{R}\right)^k\sum_{k=n}^{p(k_1,n)}{}^*(-1)^{(n+k)/2}i^{k_1-k}\left(\frac{A}{e}\right)^kC_k^{(n+k)/2}C_{-k-1}^{k_1-k};$$

$$\mu^*_{k_1,n}=(-1)^{k_1}\rho^{2n}\mu_{k_1,n}.$$

The upper limits of summation are:

$$p(k_1,n)=\begin{cases}k_1, & \text{if } k_1 \text{ and } n \text{ have the same parity;}\\ k_1-1, & \text{if } k_1 \text{ and } n \text{ have different parity;}\end{cases}$$

$$E(k_1,m)=k_1,\quad \text{if } k_1\le m;\qquad E(k_1,m)=m,\quad \text{if } k_1>m.$$

In the first case $\mu_{k_1,n}$ are real quantities, and in the second purely imaginary. f_m are real constants if m is even, and imaginary if m is odd. From the formulas for determining the coefficients $q_{k_1,m}$ and $q'_{k_1,m}$ it also follows that the latter are real if k_1 and m take values of the same parity, and purely imaginary if k_1 and m take values of different parity.

Thus the system of equations (17) splits into two: 1) a homogeneous system with respect to the unknowns $\operatorname{Re}\beta_{k_1}$ $(k_1=1,3,5,...)$ and $\operatorname{Im}\beta_{k_1}$ $(k_1=2,4,6,...)$; owing to the uniqueness of the solution they must be put equal to zero, consequently, β_{k_1} $(k_1=1,3,5,...)$ are purely imaginary, and β_{k_1} $(k_1=2,4,6,...)$ real; 2) a nonhomogeneous system

$$\sum_{k_1=1}^{\infty}A_{k_1,m}x_{k_1}=f_m\qquad(m=1,2,3,...);$$

$$A_{k_1,m}=i\left(q_{k_1,m}-q'_{k_1,m}\right);\quad x_{k_1}=\operatorname{Im}\beta_{k_1}\qquad(k_1=1,3,...); \tag{18}$$

$$A_{k_1,m}=q_{k_1,m}+q'_{k_1,m};\quad x_{k_1}=\operatorname{Re}\beta_{k_1}\qquad(k_1=2,4,...).$$

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For $e/R = 0.375,\ \rho = \sqrt{1.5}$ and $A/e = 4\sqrt{6}/25$, from the system of equations (18), by the method of successive approximations, the first 10 equations were solved, and it was necessary to find only 4 approximations.

On the basis of (4), (7), and (15), from (6) in the region S under consideration we shall have

$$\varphi_1(z) = \sum_{n=1}^{\infty} \left\{ (-1)^n \lambda_n \left(\frac{\rho}{2A}\right)^n \left[(z-ie) - \sqrt{(z-ie)^2+4A^2}\right]^n + \beta_n \left(\frac{z}{R}\right)^n \right\} +$$

$$+ iR^3 \left[e_1 \frac{z}{R} + e_3 \left(\frac{z}{R}\right)^3\right] + \beta_0. \tag{19}$$

The condition for the existence of a solution of the Neumann problem is satisfied, since the y-axis is an axis of symmetry.

The quantities

$$\Delta\% = \frac{\varphi_1(t) + \overline{\varphi_1(t)} - 2F_2(t)}{2F_2(t)} \cdot 100\%,$$

which characterize the degree of accuracy with which the function $\varphi_1(z)$ found satisfies the boundary condition at the characteristic points $z = iR,\ z = -iR$ of the circle, are respectively equal to $\Delta_1 = -0.1345 \cdot 10^{-1}\%$, $\Delta_2 = 0.1278 \cdot 10^{-2}\%$; thus the boundary condition is satisfied sufficiently accurately at the points of the circle, while the boundary condition on the ellipse is satisfied exactly.

The diagram of tangential stresses, accurate to the factor PR^2/I, is shown in Fig. 1.

4. Transforming by means of the Ostrogradsky-Green formula ((11,9) from (4)) and taking into account the expression of the function $\varphi_1(z)$ and (4), we shall have

$$y = \frac{D - Aei\sum_{n=1}^{\infty} D_1(n) - 2A^2 \sum_{n=1}^{\infty} D_2(n)}{E - \dfrac{iA}{2}\sum_{n=1}^{\infty} D_1(n)},$$

$$D = 2A(1+\rho^2)\left[i\frac{e}{\rho}(p_1 - (1+\rho^2)d_1) + 2A(b_2 - p_2)\right] +$$

$$+ eA^2(\rho^2 - \rho^{-2})\left[2(e_1R^2 - 3e_3e^2) + A^2\left(\frac{4+\sigma}{8}(\rho-\rho^{-1})^2 - 6e_3\right)\right] -$$

$$- \frac{1}{4}A^2e\left(1 - \frac{\sigma}{2}\right)\left[4(\rho^2-\rho^{-2})e^2 + 3A^2(1+\rho^2)(\rho^2 - \rho^{-2} - \rho^{-4} + 1)\right],$$

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$$E = [(1/4 - e_1)R^3 - x_1]R - A\rho^{-1}i[(1 + \rho^2)d_1 - p_1] + (\rho^2 - \rho^{-2})A^2\{e_1R^2 -$$

$$- 3e_3(e^2 + A^2) - \frac{1}{4}(1 - 1/2\sigma)(4e^2 + (\rho + \rho^{-1})^2A^2) - \frac{1}{8}\sigma(\rho - \rho^{-1})^2A^2\},$$

$$D_j(n) = \left(\frac{ie}{R}\right)^n (\rho^2 - \rho^{-2}) \sum_{\nu_1 = j-2}^{Q_j(n)} (-1)^{(\nu_1+1)/2} \left(\frac{iA}{e}\right)^{\nu_1} C_n^{\nu_1} C_{\nu_1}^{(\nu_1 - j)/2} \left[\beta_n - (-1)^{n+\nu_1}\overline{\beta_n}\right]$$

$$(j = 1,\, 2);$$

$$Q_1(n) = \begin{cases} n, & n \text{ odd}, \\ n-1, & n \text{ even}, \end{cases} \qquad Q_2(n) = \begin{cases} n, & n \text{ even}, \\ n-1, & n \text{ odd}. \end{cases}$$

For the example considered, $y = -0.118R$. At the center of bending k let us apply two equal and opposite forces (see Fig. 1). Then, under the action of the force shown by the dotted line, the bar will bend and twist under the couple whose moment is $M = Py$. Torsion is absent if the force is applied at the center of bending. In the example considered, the tangential stresses under torsion range within $0.09 \div 0.246$ of the tangential stresses under bending.

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Note: Figure translations are in progress. See original paper for figures.

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