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Abstract

Full Text

MATHEMATICS

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ON CHEBYSHEV APPROXIMATION IN A HILBERT RING

(Presented by Academician N. N. Bogolyubov, 6 IX 1957)

1. Let $\varphi(q)$ be a function continuous on the compact set Q , with values in an arbitrary Hilbert ring H (see, for example, (1)). The problem of Chebyshev approximation of a function $f(q)$, continuous on Q with values in H , by means of functions $a\varphi(q)$, where $a \in H$, consists in finding such an $a^{(0)} \in H$ that the function $a^{(0)}\varphi(q)$ deviates least on Q from $f(q)$, i.e., such that

$$\max_{q \in Q} \|a^{(0)}\varphi(q) - f(q)\| = \inf_{a \in H} \max_{q \in Q} \|a\varphi(q) - f(q)\|.$$

The case of a finite-dimensional ring was considered in (2).

In particular, if H is the ring of all complex matrices $a = \{a_{\alpha\beta}\}$ for which

$$\sum_{\alpha,\beta} |a_{\alpha\beta}|^2 < \infty \quad \text{and} \quad \|a\|^2 = \sum_{\alpha,\beta} |a_{\alpha\beta}|^2,$$

then the preceding problem is the problem of Chebyshev approximation of a matrix-function $f(q) = \{f_{\alpha\beta}(q)\}$, continuous on Q , by means of the product of matrices

$$a\varphi(q) = \{a_{\alpha\beta}\} \cdot \{\varphi_{\gamma\beta}(q)\},$$

where $\varphi(q) = \{\varphi_{\gamma\beta}(q)\}$ is a fixed matrix-function continuous on Q , and $a = \{a_{\alpha\beta}\}$ is an arbitrary numerical matrix from H .

The classical problem of Chebyshev approximation of a numerical function $f_0(q)$, continuous on Q , by a polynomial

$$\sum_{k=1}^n \xi_k \varphi_k(q),$$

where $\varphi_1(q), \dots, \varphi_n(q)$ are fixed numerical functions continuous on Q , is covered by the preceding problem when the matrix $\varphi(q)$ is a matrix of the form

$$\varphi(q) = \begin{pmatrix} \varphi_1(q) & 0 & \dots & 0 \\ \varphi_2(q) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_n(q) & 0 & \dots & 0 \end{pmatrix},$$

and the matrix $f(q)$ is a matrix of the form

$$f(q) = \begin{pmatrix} f_0(q) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

since

$$\inf_{a \in H} \max_{q \in Q} \|a\varphi(q) - f(q)\| = \inf_{\xi_k} \max_{q \in Q} \left\| \sum_{k=1}^n \xi_k \varphi_k(q) - f(q) \right\|.$$

We note that the infimum on the left-hand side is attained on a matrix all of whose rows, beginning with the second, are zero.

The problem of Chebyshev approximation in an arbitrary Hilbert ring H , in turn, is a special case of the following general problem considered in ^(3,4).

An operator-function $F(q)$ is given, which for each $q \in Q$ is a linear operator acting in the Hilbert space H , and for each fixed $x \in H$ the function $F(q)x$ is continuous on Q . It is required to find a vector $x_0 \in H$ such that

$$\max_{q \in Q} \|F(q)x_0 - f(q)\| = \inf_{x \in H} \max_{q \in H} \|F(q)x - f(q)\|.$$

In the case under consideration, $F(q)$ for each $q \in Q$ is the operator of multiplication on the right by $\varphi(q)$: $F(q)a = a\varphi(q)$.

2. As follows from ⁽³⁾ (see ⁽⁴⁾), in order that for every function $f(q)$, continuous on Q with values in H , there exist a least-deviating function, it is necessary and sufficient that the condition

$$\max_{q \in Q} \|a\varphi(q)\| \geq m\|a\| \quad \text{for all } a \in S, \tag{a}$$

be satisfied, where S is the orthogonal complement in H to the subspace T of vectors a for which $a\varphi(q) \equiv \theta$ on θ , and m is a positive constant.

Obviously, T is a closed left ideal in H , and S , as the orthogonal complement to T , is also a closed left ideal.

Choose in S and T , respectively, a maximal system of mutually orthogonal irreducible Hermitian idempotents. The union of these two systems gives a maximal system of mutually orthogonal irreducible Hermitian idempotents $\{p_\alpha\}$ for all of H . Let α' denote the indices of those idempotents of the system $\{p_\alpha\}$ which are contained in S . Then, obviously,

$$\varphi(q) = \sum_{\alpha'} p_\alpha \varphi(q).$$

The orthogonal sum

$$\sum_{\alpha'}^{\oplus} p_\alpha H$$

is the least closed right ideal containing all values $\varphi(q)$, when q ranges over the compact set Q .

The following theorem establishes the structure which the function $\varphi(q)$ must have in order that the preceding existence condition (a) be satisfied.

Theorem 1. In order that for every function $f(q)$, continuous on Q with values in H , there exist a function $a^{(0)}\varphi(q)$ least-deviating from it, it is necessary and sufficient that the least closed right ideal containing the set of all values of the function $\varphi(q)$ be the orthogonal sum of only a finite number of certain minimal right ideals p_1H, \dots, p_kH of the ring H :

$$\varphi(q) \in p_1H \oplus \dots \oplus p_kH \quad \text{for all } q \in Q,$$

or, what is the same,

$$S = H \ominus T = Hp_1 \oplus \dots \oplus Hp_k,$$

where p_1, \dots, p_k are irreducible Hermitian idempotents.

3. The following theorem gives a necessary and sufficient condition for uniqueness of the function of least deviation. It need only be noted that, since $a\varphi(q) \equiv \theta$ when $a \in T$, the question of uniqueness of the function of least deviation reduces to the question of uniqueness of the vector $a^{(0)} \in S$ for which $a^{(0)}\varphi(q)$ is the function of least deviation.

Theorem 2. Suppose the function $\varphi(q)$ satisfies the condition of Theorem 1. Then, in order that for every function $f(q)$, continuous on Q with values in H , there exist a unique function least-deviating from it, it is necessary and sufficient that, for every $a \in S$, $a \neq \theta$, the equation $a\varphi(q) = \theta$ have no roots on Q .

Remark. The characteristic property of the function $a^{(0)}\varphi(q)$ least deviating from $f(q)$, as is easily seen from (3), consists in the fact that, for every $b \in H$, the inequality

$$\min_{q \in M(a^{(0)}, f)} \operatorname{Re}(a^{(0)}\varphi(q) - f(q), b\varphi(p)) \leq 0$$

must hold, where $M(a^{(0)}, f)$ is the set of those points of Q at which

$$\max_{q \in Q} \|a^{(0)}\varphi(q) - f(q)\|$$

is attained.

4. Let us illustrate the preceding two theorems by an example in which H is a simple Hilbert ring.

Theorem 1 asserts in this case that, in an arbitrary orthonormal basis $\{p_{\alpha\beta}\}$, constructed from the above-mentioned maximal system of idempotents $\{p_{\alpha}\}$, a function $\varphi(q)$ satisfying condition (a) must have the form

$$\varphi(q) = \sum_{\beta} (\varphi_{1\beta}(q)p_{1\beta} + \dots + \varphi_{k\beta}(q)p_{k\beta}),$$

where $\varphi_{1\beta}(q), \dots, \varphi_{k\beta}(q)$ are numerical continuous functions on Q , while each vector $a \in S$ has the form

$$a = \sum_{\alpha} (a_{\alpha 1}p_{\alpha 1} + \dots + a_{\alpha k}p_{\alpha k}).$$

By virtue of the isomorphism of H to the ring of all complex matrices $a = \{a_{\alpha\beta}\}$ for which

$$\sum_{\alpha, \beta} |a_{\alpha\beta}|^2 < \infty,$$

the function $\varphi(q)$ corresponds to a matrix in which only the first k rows are nonzero, while an element $a \in S$ corresponds to a matrix in which the elements different from zero may occur only in the first k columns.

Theorem 2, in turn, asserts that for uniqueness of the function of least deviation (for every approximated function continuous on Q with values in H) it is necessary and sufficient that, for every $q \in Q$, the rank of the matrix corresponding to the function $\varphi(q)$ be equal to k .

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REFERENCES

1. M. A. Naimark, *Normed Rings*, Moscow, 1956.
2. S. I. Zukhovitskii, M. G. Krein, *Uspekhi Mat. Nauk*, 5, no. 1 (35), 217 (1950).
3. S. I. Zukhovitskii, *Mat. Sbornik*, 37 (79), no. 1, 3 (1955).
4. S. I. Zukhovitskii, G. I. Eskin, *Dokl. Akad. Nauk SSSR*, 116, no. 5 (1957).

Note: Figure translations are in progress. See original paper for figures.

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