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Abstract

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MATHEMATICS

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THE OPERATOR HELLINGER INTEGRAL AND SOME OF ITS APPLICATIONS

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1. Let H be a unitary space, and let $H_1 \oplus H_2$ be some fixed orthogonal decomposition of it. To a linear operator A, mapping H into H, there corresponds the matrix

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{1}$$

where A_{ij} is an operator mapping H_j into H_i, determined by the bilinear functional

$$(A_{ij} f, g) = (Af, g) \qquad (f \in H_j,\ g \in H_i;\ i, j = 1, 2). \tag{2}$$

The operator A will be Hermitian if and only if $A_{ij} = A^*_{ji}$. The operator A is nonnegative if and only if the operator A_{22} is nonnegative and the conditions

$$|(A_{12} f, g)|^2 \leq (A_{11} g, g)(A_{22} f, f) \qquad (g \in H_1,\ f \in H_2) \tag{3}$$

are fulfilled.

From condition (3) it follows that

$$(A_{12} f, A_{12} f) \leq C(A_{22} f, f) \qquad (f \in H_2), \tag{4}$$

where $C = \|A_{11}\|$.

Definition. A collection of operators A_{12}, A_{21}, A_{22}, where A_{22} is a nonnegative operator in H_2, A_{12} acts from H_2 into H_1, and $A_{21} = A^*_{12}$, will be called a **system of positive type** (s.p.t.) if, for some constant C, condition (4) is fulfilled. A system of positive type will be written in the form

$$\mathfrak{A} = \begin{pmatrix} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \tag{5}$$

If A_{12}, A_{21}, A_{22} are elements of the matrix (1) of an operator A, then the system (5) will be denoted by $\mathfrak{A}_A$. With a s.p.t. (5) we associate the operator $\omega_{\mathfrak{A}}$, defined on $R(A_{22}^{1/2})$ by the equalities

$$z = A_{22}^{1/2} f, \qquad \omega_{\mathfrak{A}} z = A_{12} f \qquad (f \in H_2). \tag{6}$$

This operator is bounded and can be extended by continuity to the subspace $\overline{R(A_{22}^{1/2})}$. The operator $\omega_{\mathfrak{A}} \omega_{\mathfrak{A}}^*$, which we shall denote by $\widehat{\mathfrak{A}}$, is a Hermitian operator in H_1. (If A_{22} has a bounded inverse, then $\widehat{\mathfrak{A}} = A_{12} A_{22}^{-1} A_{21}$.)

Theorem 1. In order that the operator A be nonnegative, it is necessary and sufficient that $\mathfrak{A}_A$ be a s.p.t. and that the condition

$$A_{11} \geq \widehat{\mathfrak{A}}_A \tag{7}$$

be fulfilled.

Corollary. The system $\mathfrak{A}_A$ will be a s.p.t. if and only if the class of operators C satisfying the conditions

$$C(H_2) = 0, \qquad C \leq A. \tag{8}$$

is nonempty. In this class there is a maximal operator, which we shall denote by $\widetilde{A}$. Obviously,

$$\widetilde{A} = \begin{pmatrix} A_{11} - \mathfrak{A}_A & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

For nonnegative operators the existence of $\widetilde{A}$ was established in (1).

2. Let (X, S) be some measurable space*; let H' and H'' be two unitary spaces. A function $F(M)$ that assigns to each $M \in S$ a certain linear operator mapping H' into H'' will be called **completely additive** if, for every finite or countable system of pairwise disjoint sets $M_k \in S$, the condition

$$F\left(\bigcup_k M_k\right) = \sum_k F(M_k) \tag{10}$$

is fulfilled.

If the number of summands is infinite, then the series on the right-hand side of (10) converges in the weak sense, and its sum does not depend on the order of the summands. If the values of the function $F(M)$ are nonnegative operators, then it is called an **operator measure**. Let $F_{12}(M), F_{21}(M), F_{22}(M)$ be completely additive operator functions defined on S, where for each $M \in S$, $F_{22}(M)$ is a nonnegative operator in H_2; $F_{12}(M)$ maps H_2 into H_1; $F_{21}(M) = [F_{12}(M)]^*$ maps H_1 into H_2. We shall write the system of these operators in the form

$$\mathfrak{F}(M) = \begin{pmatrix} \cdot & F_{12}(M) \\ F_{21}(M) & F_{22}(M) \end{pmatrix}. \tag{11}$$

Assume that for any $M \in S$, $\mathfrak{F}(M)$ is a s.p.t. For an arbitrary partition

$$M = M_1 \cup M_2 \cup \cdots \cup M_n \tag{12}$$

of a certain fixed set $M \in S$ into a finite number of pairwise disjoint summands $M_k \in S$, we form the operator

$$\sum_{k=1}^{n} \widehat{\mathfrak{F}}(M_k). \tag{13}$$

If the set of all such operators (corresponding to all possible partitions of M) is bounded above, then we shall say that the function (11) is integrable on the set M. By the equality

$$\alpha(g) = \sup \sum_{k=1}^{n} (\widehat{\mathfrak{F}}(M_k)g, g) \qquad (g \in H_1), \tag{14}$$

where the sup is taken over all partitions (12), a certain quadratic functional $\alpha(g)$ is defined. The corresponding operator $\widehat{\mathfrak{F}}(M)$ will be called the **Hellinger integral** of the function (11), and we shall write it in the form

$$\widehat{\mathfrak{F}}(M) = \int_M dF_{12}\,(dF_{22})^{-1} dF_{21}. \tag{15}$$

Theorem 2. $\widehat{\mathfrak{F}}(M)$ is a completely additive function and, consequently, is an operator measure.

* In this section we shall use the terminology and notation of the monograph (2).

Theorem 3. A completely additive function

$$F(M) = \begin{pmatrix} F_{11}(M) & F_{12}(M) \\ F_{21}(M) & F_{22}(M) \end{pmatrix}, \tag{16}$$

whose values are Hermitian operators, is a measure if and only if the corresponding system (11) is integrable on every $M \in S$ and the inequality

$$F_{11}(M) \geq \int_M dF_{12}(dF_{22})^{-1}dF_{21} \qquad (M \in S) \tag{17}$$

is satisfied.

Let us order the set of operator measures by putting $F' < F''$ if, for every $M \in S$, the condition $F'(M) \leq F''(M)$ is fulfilled. Then the following proposition can be stated.

Corollary. In the class of all measures F' satisfying the conditions

$$F'(M)f = 0 \quad (f \in H_2,\ M \in S); \qquad F' \leq F,$$

there exists a maximal $\widetilde{F}(M)$, namely

$$\widetilde{F}(M) = \begin{pmatrix} F_{11}(M) - \mathfrak{F}(M) & 0 \\ 0 & 0 \end{pmatrix}.$$

3. Let

$$\mathfrak{A}(\zeta) = \begin{pmatrix} & A_{12}(\zeta) \\ A_{21}(\zeta) & A_{22}(\zeta) \end{pmatrix} \tag{18}$$

be a p.s.d. function whose elements are harmonic in the domain G functions of the parameter $\zeta = x + yi$. Then $\mathfrak{A}(\zeta)$ is a subharmonic* function in the domain G. Its least harmonic majorant (if it exists) will be denoted by $\widehat{\mathfrak{A}}(\zeta)$. If

$$A(\zeta) = \begin{pmatrix} A_{11}(\zeta) & A_{12}(\zeta) \\ A_{21}(\zeta) & A_{22}(\zeta) \end{pmatrix} \tag{19}$$

is a function harmonic in G for which the corresponding system (18) is p.s.d., then the function

$$\widetilde{A}(\zeta) = \begin{pmatrix} A_{11}(\zeta) - \widehat{\mathfrak{A}}(\zeta) & 0 \\ 0 & 0 \end{pmatrix}$$

is maximal among all harmonic functions $C(\zeta)$ satisfying the conditions

$$C(\zeta)f = 0 \quad (f \in H_2,\ \zeta \in G), \qquad C(\zeta) \leq A(\zeta) \quad (\zeta \in G).$$

Let the system (11) be defined in the class of Borel sets of the segment $[0, 2\pi]$, and let the system (18), harmonic for $|\zeta| < 1$, be connected with it by the Poisson integral

$$A_{ij}(re^{i\theta}) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \theta)}\, dF_{ij}(t). \tag{20}$$

Then the following theorem is valid.

Theorem 4. The subharmonic function $\mathfrak{A}(\zeta)$ for $|\zeta| < 1$ has a harmonic majorant if and only if the system (11) integrable on the segment $[0, 2\pi]$. In this case

$$\mathfrak{A}(re^{i\theta}) = \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(t - \theta)}\, d\widehat{\mathfrak{F}}(t).$$

4. Let A be a self-adjoint operator (in general unbounded), and let E_t be its spectral function, which, as usual, we extend to a spectral measure $E(M)$. Denote by H_0 the invariant subspace generated by the subspace H_2.

Theorem 5. $\widetilde{E}(M)$ is determined by the condition

$$\widetilde{E}(M)f = \begin{cases} 0, & \text{if } f \in H_0, \\ E(M)f, & \text{if } f \perp H_0. \end{cases}$$

Now let D be some generating subspace of the operator A; let $F(M)$ be an operator measure whose values are Hermitian operators in D, determined by the bilinear functional

$$(F(M)f, g) = (E(M)f, g) \qquad (f, g \in D).$$

To the decomposition $D = D_1 \oplus D_2$ there corresponds a representation of $F(M)$ in the form (16).

Theorem 6. In order that D_2 be a generating subspace, it is necessary and sufficient that $\widetilde{F}(M) \equiv 0$.

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* An operator function $A(\zeta)$, taking Hermitian values, will be called **subharmonic** if, for every f, the function $(A(\zeta)f, f)$ is subharmonic.

Let U be a unitary operator in H; let D be its generating subspace; let $A(\zeta)$ be the harmonic function for $|\zeta| < 1$ determined by the quadratic functional

$$(A(\zeta)f, f) = Re\,((U + \zeta I)(U - \zeta I)^{-1}f, f) \qquad (f \in D).$$

If $D = D_1 \oplus D_2$ is some orthogonal decomposition of D,

$$A(\zeta) = \begin{pmatrix} A_{11}(\zeta) & A_{12}(\zeta) \\ A_{21}(\zeta) & A_{22}(\zeta) \end{pmatrix}$$

is the corresponding matrix representation of $A(\zeta)$, then the following theorem holds.

Theorem 7. In order that D_2 be a generating subspace of the operator U, it is necessary and sufficient that $\widetilde{A}(\zeta) \equiv 0$ ($|\zeta| < 1$).

5. Theorems 6 and 7 can be applied to establish criteria for simplicity of isometric extensions of isometric operators. We shall give some results.

a) Let V_1 and V_2 be simple (3) isometric operators with defect indices $(1, 1)$; let $w_1(\zeta)$ and $w_2(\zeta)$ be their characteristic functions. Taking as the coupling matrix (4) the matrix $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, we obtain an isometric operator V with defect indices $(1, 1)$ and characteristic function $w_1(\zeta)w_2(\zeta)$ (4). The operator V is simple if and only if, at almost every point e^{it} of the unit circle, either $|w_1(e^{it})|$ or $|w_2(e^{it})|$ is equal to one.

b) Let V be a simple isometric operator with defect indices $(1, 2)$ and characteristic function $(w_1(\zeta), w_2(\zeta))$. Extend V to an operator with defect indices $(0, 1)$, mapping the first defect subspace onto the first axis of the second. The resulting operator will be simple if and only if

$$\int_0^{2\pi} \left| \frac{w_2(e^{it})}{1 - w_1(e^{it})} \right|^2 dt = 2\pi.$$

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