MATHEMATICS

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Abstract

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A DIFFERENTIAL EQUATION WITH AN ABSTRACT ELLIPTIC OPERATOR IN HILBERT SPACE

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1. Consider the differential equation

$$\frac{dv}{dt} + Av = 0, \tag{1}$$

where A is a certain unbounded operator with everywhere dense domain of definition $D(A)$, acting in a Hilbert space H; $v = v(t)$ is the unknown function with values in H, satisfying the prescribed initial condition

$$v(0) = v_0 \in D(A). \tag{2}$$

The function $v(t)$ is a solution of equation (1) on $[0, T]$ if, for $t \in [0, T]$, it has a strong derivative dv/dt, its values belong to the domain of definition $D(A)$ of the operator A, and they satisfy equation (1).

Definition. We shall say that problem (1)-(2) is **well posed** on the segment $[0, T]$ if: 1) a solution of the problem exists for every $v_0 \in D(A)$; 2) the solution is unique; 3) the solution depends continuously on the initial data uniformly in t on $[0, T]$, i.e., if $v_n(0) \in D(A)$ and $v_n(0) \to 0$, then

$$\max_{0 \le t \le T} \|v_n(t)\| \to 0$$

as $n \to \infty$.

It is easy to see that a problem well posed on the segment $[0, T]$ will also be well posed on any larger segment; therefore, in the definition of well-posedness we shall omit mention of the segment.

Equation (1) was studied by semigroup-theory methods by Hille ([1]), Yosida ([2]), Phillips ([3]), and others. The naturalness of applying semigroups in the study of such equations follows from the following theorem.

Theorem 1. *In order that problem (1)-(2) be well posed, it is necessary that the operator $-A$ admit a closure $-\overline{A}$ which is the infinitesimal generator of a strongly continuous semigroup $U(t)$ of bounded operators. If A is closed, then the condition is sufficient. The solution of problem (1)-(2) is given by the formula*

$$v(t) = U(t)v_0. \tag{3}$$

In proving the theorem we use Hille' s method ([4]).

2. Everywhere below, by A we shall mean a positive-definite self-adjoint operator acting in H. Suppose, for definiteness, that

$$\inf_{v \in D(A)} \frac{(Av, v)}{\|v\|^2} = 1. \tag{4}$$

The semigroup generated by equation (1) with such an operator can be represented in the form

$$U(t) = e^{-At}.$$

In addition to this semigroup, we shall consider the semigroup $A^{-\alpha}$ of negative powers of the operator A. These two semigroups are connected with each other by the Mellin transform

$$\Gamma(\alpha)A^{-\alpha} = \int_0^\infty t^{\alpha-1} e^{-At}\, dt. \tag{5}$$

Let us note that, with the aid of the spectral decomposition of the operator A, one easily obtains the estimate ([5])

$$\|A^\alpha e^{-tA}\| = \begin{cases} \left(\dfrac{\alpha}{e}\right)^\alpha \dfrac{1}{t^\alpha}, & \text{for } t \leq \alpha, \\ e^{-t}, & \text{for } t \geq \alpha. \end{cases} \tag{6}$$

Definition. We shall call the operator B an **operator of fractional order with respect to the positive-definite operator** A, if there exist positive constants $\gamma < 1$ and K_γ such that

$$\|Bv\| \leq K_\gamma \|A^\gamma v\| \tag{7}$$

for every $v \in D(A)$. The greatest lower bound of the numbers γ for which inequality (7) holds will be called the **order** of the operator B.

In practice, finding the fractional powers of concrete operators, for example differential ones, is very difficult. Therefore, for determining that the operator B is an operator of fractional order with respect to A, Theorem 2 is useful.

Theorem 2. In order that the operator B be of fractional order α with respect to A, it is necessary, and if B admits closure, then also sufficient, that for all $v \in D(A)$, $\gamma > \alpha$, and sufficiently small δ, the inequality

$$\|Bv\| \leq \delta^{1-\gamma}\|Av\| + \frac{K}{\delta^\gamma}\|v\|, \tag{8}$$

hold, where K does not depend on δ and v.

Proof. Necessity. By means of Hölder' s inequality, one establishes the inequality

$$\|A^\gamma v\| \leq \|Av\|^\gamma \|v\|^{1-\gamma} \qquad (v \in D(A)).$$

Then from (7) it follows that

$$\|Bv\| \leq K_\gamma \|A^\gamma v\| \leq K_\gamma \|Av\|^\gamma \|v\|^{1-\gamma} \leq \left\{\frac{\delta^{1-\gamma}}{\gamma}\|Av\|\right\}^\gamma \left\{\frac{\gamma^{\gamma/(1-\gamma)}K_\gamma^{1/(1-\gamma)}}{\delta^\gamma}\|v\|\right\}^{1-\gamma}.$$

Applying Minkowski' s inequality to the right-hand side with $p = 1/\gamma$ and $q = 1/(1-\gamma)$, we obtain inequality (8) with $K = (1-\gamma)\gamma^{\gamma/(1-\gamma)}K_\gamma^{1/(1-\gamma)}$.

Sufficiency. We shall use (5) for $A^{-\gamma}$ $(\gamma > \alpha)$. Then, for $w \in D(A^{1-\gamma})$,

$$\|BA^{-\gamma}w\| \leq \frac{1}{\Gamma(\gamma)}\int_0^{\delta_0} t^{\gamma-1}\|BU(t)w\|\,dt + \frac{1}{\Gamma(\gamma)}\int_{\delta_0}^\infty t^{\gamma-1}\|BU(t)w\|\,dt.$$

Suppose that (8) is satisfied for $\gamma > \alpha$ and $\delta \leq \delta_0$. Put $\gamma_1 = (\alpha+\gamma)/2$. To the first integral we apply (8), putting $\delta = t$ and $\gamma = \gamma_1$, and to the second, putting $\delta = \delta_0$ and $\gamma = \gamma_1$. Then

$$\|BA^{-\alpha}w\| \leq \frac{1}{\Gamma(\gamma)}\left\{\int_0^{\delta_0} t^{\gamma-\gamma_1}\|AU(t)\|\,dt + K\int_0^{\delta_0} t^{\gamma-\gamma_0-1}\|U(t)\|\,dt \right.$$
$$\left. + \int_{\delta_0}^\infty t^{\gamma-1}\left[\delta_0^{1-\gamma_1}\|AU(t)\| + \frac{K}{\delta_0^{\gamma_1}}\|U(t)\|\right]dt\right\}\|w\|.$$

In view of estimate (6), all integrals exist; therefore $\|BA^{-\gamma}w\| \leqslant K_\gamma\|w\|$ for $w \in D(A^{1-\gamma})$. Hence (7) follows. The theorem is proved.

3. **Definition.** We shall call the operator S **elliptic** if it is the sum of a positive-definite self-adjoint operator A and an operator B of fractional order with respect to A: $S = A + B$. It is not difficult to show that an elliptic operator considered on $D(A)$ is closed.

Consider the equation

$$\frac{dv}{dt} + (A + B)v = 0. \tag{9}$$

Theorem 3. *The problem (9)–(2) with an elliptic operator is well posed.*

For the proof it is necessary to show the existence of a solution, uniqueness, and continuous dependence on the initial data. We describe the scheme of the proof.

Uniqueness. Multiply (9) scalarly by Av. We obtain

$$\mathrm{Re}\left(\frac{dv}{dt}, Av\right) = -(Av, Av) - \mathrm{Re}(Bv, Av) \leqslant$$

$$\leqslant -(Av, Av) + \|Bv\|\|Av\| \leqslant -\frac{1}{2}\|Av\|^2 + \frac{1}{2}\|Bv\|^2.$$

Applying Theorem 2, then

$$\mathrm{Re}\left(\frac{dv}{dt}, Av\right) \leqslant -\frac{1}{2}\|Av\|^2 + \delta^{2(1-\gamma)}\|Av\|^2 + \frac{K^2}{\delta^{2\gamma}}\|v\|^2.$$

Choosing δ so that $\delta^{2(1-\gamma)} < 1/2$, we have

$$\mathrm{Re}\left(\frac{dv}{dt}, Av\right) \leqslant \frac{K^2}{\delta^{2\gamma}}\|v\|^2 \leqslant \frac{K^2}{\delta^{2\gamma}}\|A^{1/2}v\|^2.$$

By simple reasoning, from this we obtain the inequality

$$\|A^{1/2}v(t)\| \leqslant \|A^{1/2}v(0)\|e^{\frac{K^2}{\delta^{2\gamma}}t},$$

from which the uniqueness of the solution of the problem (9)–(2) follows.

Existence. We introduce into consideration the integral equations

$$v(t) = e^{-At}v_0 - \int_0^t e^{-(t-s)A}Bv(s)\,ds, \tag{10}$$

$$w(t) = A^{\gamma}e^{-At}v_0 - \int_0^t A^{\gamma}e^{-(t-s)A}BA^{-\gamma}w(s)\,ds. \tag{11}$$

With the aid of the results of work ([5]), the following lemma is proved.

Lemma. *Every continuous solution $w(t)$ of equation (11), by the formula $v(t) = A^{-\gamma}w(t)$, generates a solution of problem (9)–(2).*

A continuous solution of equation (11) is constructed by the method of successive approximations. We note that the kernel of equation (11), in view of (6) and (7), has a summable singularity of the form

$$\|A^{\gamma}e^{-(t-s)A}BA^{-\alpha}\| \leqslant \left(\frac{\gamma}{e}\right)^{\gamma} K_{\gamma}\frac{1}{(t-s)^{\gamma}}.$$

Continuous dependence on the initial data is proved in two stages. First, from equation (11) on a sufficiently small segment $[0, t_1]$ one obtains an estimate for $w(t)$ of the form

$$\|A^{\gamma}v(t)\| = \|w(t)\| \leqslant 2\left(\frac{\gamma}{e}\right)^{\gamma}\frac{\|v_0\|}{t^{\gamma}}.$$

From this estimate, with the aid of the integral equation (10), it follows that, for $0 \leqslant t \leqslant t_1$, $\|v(t)\| \leqslant K_2\|v_0\|$. The theorem is proved.

Corollary 1. For every elliptic operator S, the operator $-S$ is the infinitesimal generator of a strongly continuous semigroup.

Corollary 2. The spectrum of an elliptic operator lies in some half-plane $\operatorname{Re}\lambda \geqslant \omega$.

Denote by $U_S(t)$ the semigroup generated by equation (9). By means of a detailed study of the integral equations (10) and (11), one can obtain for the semigroup $U_S(t)$ estimates analogous to the first estimate in (6). In particular, there exist k_1 and $T > 0$ such that

$$\|S^m U_S(t)\| \leqslant \frac{k_1^m m^m}{t^m} \tag{12}$$

for $0 < t \leqslant mT$. Estimate (12) makes it possible to prove Theorem 4.

Theorem 4. *For every $v \in H$, the function $U_S(t)v$ satisfies equation (9) for $t > 0$. All solutions $U_S(t)v$ of equation (9) are analytic inside a certain angle $|\arg t| < \varphi_0$ (φ_0 does not depend on v).*

Let us note a property of one particular type of elliptic operators. Suppose that the operator B satisfies one of the following two conditions: a) the operator B is of fractional order and the operator $BA^{-1/2}$ is bounded, or b) the operators B and B^* are operators of fractional order. Then the inequality

$$((A+B)v, v) \geqslant C_1(Av, v) - C_2(v, v), \tag{13}$$

holds, where C_1 and $C_2 > 0$ do not depend on v. In this case the elliptic operator S satisfies the Hille–Yosida conditions ([1]). In deriving estimate (13), the important Heinz inequality ([6]) is used essentially.

4. One may consider the equation with a variable elliptic operator $S(t)$

$$\frac{dv}{dt} + [A(t) + B(t)]v = f(t).$$

If $A(t)$ satisfies the conditions of work ([5]), and $B(t)$ is strongly continuous on $D(A)$, then all the main results of work ([5]) carry over to this equation.

5. In order to clarify the range of possible applications of the results presented, let us consider, for example, a self-adjoint positive-definite differential operator L_{2m} of order $2m$ in a bounded domain G of n-dimensional space. By A we shall denote the self-adjoint operator generated by the operator L_{2m} on functions from $\mathcal{L}_2(G)$ satisfying homogeneous self-adjoint boundary conditions, for example, the first boundary-value problem. Then, under sufficiently general conditions, it turns out that differential operators of order $k < 2m$ are operators of fractional order $k/2m$ relative to the operator A. Adding them to the operator L_{2m} leads to elliptic operators in our sense. Thus, differential operators of elliptic type are included in our considerations. The proof of the fact noted above is based on Theorem 2, while the verification of its conditions is carried out with the help of inequalities of the type of the Ladyzhenskaya-Guseva inequality ([7,8]) and Nirenberg' s inequality ([9]).

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Note: Figure translations are in progress. See original paper for figures.