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Abstract

Full Text

MATHEMATICS

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ILL-POSEDNESS OF THE DIRICHLET PROBLEM FOR EQUATIONS OF MIXED TYPE IN MIXED DOMAINS

As a model equation of mixed type we take the equation of M. A. Lavrent' ev

$$U_{xx} + \operatorname{sgn} y \cdot U_{yy} = 0. \tag{1}$$

Denote by D a simply connected mixed domain bounded by a smooth Jordan arc σ, lying in the upper half-plane and with endpoints at the points $A(0,0)$, $B(1,0)$, and by curves of continuous curvature issuing from these points,

$$L: \ y = -\gamma(x), \qquad L_1: \ y = -\gamma_1(x),$$

satisfying the conditions

$$\gamma > 0, \qquad \gamma_1 > 0, \qquad 0 < \gamma'(x) < 1, \qquad 0 < -\gamma_1'(x) < 1. \tag{2}$$

Let $C[x_1, -\gamma(x_1)]$ be the point of intersection of the curves L and L_1. From the point $E(x_0, 0)$, where $x_1 - \gamma(x_1) \le x_0 \le x_1 + \gamma(x_1)$, draw the characteristics $EB_2: \ y = x - x_0$ and $EB_1: \ y = x_0 - x$, where B_2 and B_1 are the points of intersection of the indicated characteristics with the curves L and L_1. Denote by L_2 and L_3 the arcs AB_2 and BB_1 of the curves L and L_1, respectively.

For equation (1) in the domain D the Dirichlet problem is ill-posed independently of the size and form of the hyperbolic part of the domain D.

This follows from the fact that the following mixed problem always has, and moreover has uniquely, a solution.

It is required to find a function $U(x,y)$ with the following properties: 1) $U(x,y)$ is a solution of equation (1) in the domain D for $y \ne 0$, $y \ne x - x_0$, $y \ne x_0 - x$; 2) it is continuous in the closed domain $\overline{D}$; 3) the partial derivatives U_x and U_y are continuous in the closed domain D everywhere except, possibly, at the points A, B and on the segments EB_2, EB_1, near which they may become infinite of order less than 1/2; 4) on the lines σ, L_2, and L_3 the function $U(x,y)$ assumes the prescribed values

$$U\Big|_{\sigma} = \psi_1, \qquad U\Big|_{L_2} = \psi_2, \qquad U\Big|_{L_3} = \psi_3, \tag{3}$$

where ψ_1 is a continuous function, and ψ_2 and ψ_3 are twice continuously differentiable functions.

Without loss of generality one may assume that

$$\psi_1 \equiv 0, \qquad \psi_2(0) = \psi_2'(0) = \psi_3(1) = \psi_3'(1) = 0.$$

For simplicity of computation suppose that $\gamma = \alpha x,\ \gamma_1 = -\beta x + \beta$, where α and β are constants satisfying $0 < \alpha < 1,\ 0 < \beta < 1$. In this case

$$C = C\left(\frac{\beta}{\alpha+\beta}, -\frac{\alpha\beta}{\alpha+\beta}\right), \qquad B_2 = B_2\left(\frac{x_0}{1+\alpha}, -\frac{\alpha x_0}{1+\alpha}\right), \qquad B_1 = B_1\left[\frac{x_0+\beta}{1+\beta}, \frac{\beta(x_0-1)}{1+\beta}\right].$$

The result obtained below remains valid also under general assumptions concerning L and L_1, provided conditions (2) are satisfied.

The general solution of equation (1), satisfying conditions (3), in the triangles AB_2E and EB_1B, respectively, has the form

$$U(x,y) = f(x+y) - f[\lambda(x-y)] + \psi_2\left[\frac{1+\lambda}{2}(x-y)\right], \tag{4}$$

$$U(x,y) = \varphi(x-y) - \varphi[\mu(x+y) + 1 - \mu] + \psi_3\left[\frac{(x+y)(1+\mu)+1-\mu}{2}\right], \tag{5}$$

where $f(t),\ 0 < t < x_0,\ \varphi(t),\ x_0 < t < 1$, are arbitrary twice continuously differentiable functions, and

$$\lambda = \frac{1-\alpha}{1+\alpha}, \qquad \mu = \frac{1-\beta}{1+\beta}.$$

From (4) and (5) we have

$$U_x(x,0) + U_y(x,0) = 2f_x(x), \qquad 0 < x < x_0;$$

$$U_x(x,0) - U_y(x,0) = 2\varphi_x(x), \qquad x_0 < x < 1;$$

$$U_x(x,0) - U_y(x,0) = -2f_x(\lambda x) + 2\psi_{2x}\left(\frac{1+\lambda}{2}x\right), \qquad 0 < x < x_0; \tag{6}$$

$$U_x(x,0)+U_y(x,0)=-2\varphi_x(\mu x+1-\mu)+$$

$$+2\psi_{3x}\left[\frac{x(1+\mu)+1-\mu}{2}\right],\qquad x_0<x<1. \tag{7}$$

Denote by $F_0(z)=U_0(x,y)+iV_0(x,y)$ the function analytic in the elliptic part of the domain D, whose real part is the solution of the homogeneous mixed problem.

Under the additional requirement imposed on the arc σ:

$$\omega(s)=\operatorname{Im}\left\{z(1-z)(x_0-z)\left(\frac{dy}{dx}+i\right)^2 dz\right\}_\sigma\leq 0,$$

where s is the length of the arc σ (measured from the point B), the uniqueness of the solution of the problem posed above follows from the fact that

$$I=\int_0^1 x(1-x)(x_0-x)U_{0x}U_{0y}\,dx=\int_\sigma \omega(s)\left(\frac{\partial U_0}{\partial y}\right)^2 ds=0.$$

In particular, when σ coincides with the semicircle $\sigma_0: 2z=1+e^{i\varphi}$, $\operatorname{Im} z\geq 0$, the function $\omega(s)=-1/8$.

Denoting the right-hand sides in formulas (6) and (7), respectively, by $\omega_1(x)$ and $\omega_2(x)$, we can rewrite these formulas in the form

$$\operatorname{Re}(1-i)F'(x)=\omega_1(x),\qquad 0<x<x_0;$$

$$\operatorname{Im}(1-i)F'(x)=-\omega_2(x),\qquad x_0<x<1, \tag{8}$$

where $F(z)=U(x,y)+iV(x,y)$ is a holomorphic function in the elliptic part of the domain D, whose real part is the solution of the problem posed above.

We shall restrict ourselves to considering the case where σ coincides with the semicircle σ_0, and $x_0>1/2$. A holomorphic function $F(z)$ satisfying conditions (8), whose real part is zero on σ_0, is determined by the formula

$$(1-i)F'(z) = \frac{1}{\pi i}\sqrt{\frac{z(1-z)}{(x_0-z)(\xi_0-z)}}\left\{\int_0^{x_0}\sqrt{\frac{(x_0-t)(\xi_0-t)}{t(1-t)}}\left(\frac{1}{t-z}+\frac{1-2t}{t+z-2tz}\right)\right.$$
$$\left.\times\omega_1(t)\,dt - \int_{x_0}^{1}\sqrt{\frac{(t-x_0)(\xi_0-t)}{t(1-t)}}\left(\frac{1}{t-z}+\frac{1-2t}{t+z-2tz}\right)\omega_2(t)\,dt\right\}, \tag{9}$$

where $(2x_0-1)\xi_0 = x_0$, and by the root is meant its single-valued branch in the plane cut along the segments $[0,x_0]$, $[1,\xi_0]$, positive for $0<z<x_0$.

Passing to the limit in (9) as $z\to x$ and taking the imaginary part for $0<x<x_0$, and the real part for $x_0<x<1$, we obtain

$$f_x(x)+\frac{1}{\pi}\int_0^{x_0}\sqrt{\frac{x(1-t)(x_0-t)(\xi_0-t)}{t(1-x)(x_0-x)(\xi_0-x)}}\left(\frac{1}{t-x}+\frac{1}{t+x-2tx}\right)f_t(\lambda t)\,dt = \rho_1(x), \tag{10}$$

$$\varphi_x(x)-\frac{1}{\pi}\int_{x_0}^{1}\sqrt{\frac{t(1-x)(t-x_0)(\xi_0-t)}{x(1-t)(x-x_0)(\xi_0-x)}}\left(\frac{1}{t-x}-\frac{1}{t+x-2tx}\right)$$
$$\times\varphi_t(\mu t+1-\mu)\,dt = \rho_2(x), \tag{11}$$

$$\rho_1(x) = \psi(x)+\frac{1}{\pi}\int_{x_0}^{1}\sqrt{\frac{x(1-x)(t-x_0)(\xi_0-t)}{t(1-t)(x_0-x)(\xi_0-x)}}\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)$$
$$\times\varphi_t(\mu t+1-\mu)\,dt,$$

$$\rho_2(x) = \psi(x)-\frac{1}{\pi}\int_0^{x_0}\sqrt{\frac{x(1-x)(x_0-t)(\xi_0-t)}{t(1-t)(x-x_0)(\xi_0-x)}}\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)f_t(\lambda t)\,dt,$$

$$\psi(x) = \frac{1}{\pi}\sqrt{\frac{x(1-x)}{(x_0-x)(\xi_0-x)}}\left\{\int_0^{x_0}\sqrt{\frac{(x_0-t)(\xi_0-t)}{t(1-t)}}\right.$$
$$\times\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\psi_{2t}\left(\frac{1+\lambda}{2}t\right)dt$$
$$-\int_{x_0}^{1}\sqrt{\frac{(t-x_0)(\xi_0-t)}{t(1-t)}}$$
$$\left.\times\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\psi_{3t}\left[\frac{t(1+\mu)+1-\mu}{2}\right]dt\right\}, \qquad 0<x<x_0;$$

$$\psi(x) = \frac{1}{\pi}\sqrt{\frac{x(1-x)}{(x-x_0)(\xi_0-x)}}\left\{\int_0^{x_0}\sqrt{\frac{(x_0-t)(\xi_0-t)}{t(1-t)}}\right.$$
$$\times\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\psi_{2t}\left(\frac{1+\lambda}{2}\,t\right)dt$$
$$-\int_{x_0}^{1}\sqrt{\frac{(t-x_0)(\xi_0-t)}{t(1-t)}}$$
$$\left.\times\left(\frac{1}{t-x}+\frac{1-2t}{t+x-2tx}\right)\psi_{3t}\left[\frac{t(1+\mu)+1-\mu}{2}\right]dt\right\}, \qquad x_0<x<1.$$

By a simple change of variables, equations (10) and (11), in the notation

$$\sqrt{x}\,\mu_1(x) = f_x(x)\sqrt{(1-x)(x_0-x)(\xi_0-x)},$$

$$\sqrt{x}\,h_1(x) = \rho_1(x)\sqrt{(1-x)(x_0-x)(\xi_0-x)},$$

$$\sqrt{1-x}\,\mu_2(x) = \varphi_x(x)\sqrt{x(x-x_0)(\xi_0-x)},$$

$$\sqrt{1-x}\,h_2(x) = \rho_2(x)\sqrt{x(x-x_0)(\xi_0-x)}$$

can be rewritten in the following form:

$$\mu_1(x)+\frac{1}{\pi}\int_0^{\lambda x_0}\sqrt{\frac{(\lambda-t)(\lambda x_0-t)(\lambda\xi_0-t)}{(1-t)(x_0-t)(\xi_0-t)}}\left(\frac{1}{t-\lambda x}+\frac{1}{t+\lambda x-2tx}\right)\mu_1(t)\,dt =$$

$$= h_1(x), \qquad 0<x<x_0, \tag{12}$$

$$\mu_2(x)-\frac{1}{\pi}\int_{\mu x_0+1-\mu}^{1}\sqrt{\frac{(t-1+\mu)(t-1+\mu-\mu x_0)(\mu\xi_0-t+1-\mu)}{t(t-x_0)(\xi_0-t)}}\left(\frac{1}{t-1+\mu-\mu x}\right.$$

$$\left.-\frac{1}{t-1+\mu+\mu x-2x(t-1+\mu)}\right)\mu_2(t)\,dt = h_2(x), \qquad x_0<x<1. \tag{13}$$

In view of the assumptions made, the functions $\mu_1(x)$ and $\mu_2(x)$ are naturally to be sought in Hilbert space (L^2).

We now show that the norms of the integral operators in the left-hand sides of (12) and (13), for $0 < x < \lambda x_0$, $\mu x_0 + 1 - \mu < x < 1$, are less than one. This follows from the fact that

$$\frac{1}{\pi^2}\int_0^{\lambda x_0} d\xi \int_0^{\lambda x_0} \sqrt{\frac{(\lambda - t)(\lambda x_0 - t)(\lambda \xi_0 - t)}{(1-t)(x_0-t)(\xi_0-t)}}\left(\frac{1}{t-\lambda\xi} + \frac{1}{t+\lambda\xi - 2t\xi}\right)\mu_1(t)\,dt\times$$

$$\times\int_0^{\lambda x_0} \sqrt{\frac{(\lambda - \tau)(\lambda x_0 - \tau)(\lambda \xi_0 - \tau)}{(1-\tau)(x_0-\tau)(\xi_0-\tau)}}\left(\frac{1}{\tau-\lambda\xi} + \frac{1}{\tau+\lambda\xi - 2\tau\xi}\right)\mu_1(\tau)\,d\tau \leq$$

$$\leq \frac{1}{\lambda}\int_0^{\lambda x_0} \frac{(\lambda - t)(\lambda x_0 - t)(\lambda \xi_0 - t)}{(1-t)(x_0-t)(\xi_0-t)}\mu_1^2(t)\,dt \leq \lambda^2 \int_0^{\lambda x_0} \mu_1^2(t)\,dt.$$

The validity of the second part of our assertion can be shown analogously.

Consequently, the solutions $\mu_1(x)$, $0 < x < \lambda x_0$, and $\mu_2(x)$, $\mu x_0 + 1 - \mu < x < 1$, of equations (10) and (11) exist, and they are expressed by singular integrals. Substituting the expression $f_t(\lambda t)$ obtained as a result of inverting equation (10) into the right-hand side of (11), and inverting the left-hand side of the latter equation with respect to $\varphi_x(x)$, $\mu x_0 + 1 - \mu < x < 1$, we obtain a Fredholm integral equation of the second kind for the relative unknown function φ, equivalent to the problem. Therefore, the existence of a solution of the resulting equation follows from the uniqueness of the solution of the mixed problem.

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Note: Figure translations are in progress. See original paper for figures.