TENSOR PRODUCTS OF SYSTEMS OF GROUPS AND UNIVERSAL COEFFICIENT THEOREMS FOR HOMOLOGY AND COHOMOLOGY

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Abstract

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TENSOR PRODUCTS OF SYSTEMS OF GROUPS AND UNIVERSAL COEFFICIENT THEOREMS FOR HOMOLOGY AND COHOMOLOGY

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1. The problem of finding a universal coefficient group for the spectral groups of homology or cohomology of a topological space in the sense of Čech was the subject of a paper by Steenrod ([1]). However, owing to the use of an incorrect algebraic lemma, what is actually obtained in that paper is not the formula

$$H^q(X,G) \approx H_0^q(X) \otimes G + H_0^{q+1}(X) * G, \tag{1}$$

which would mean the universality of the integral coefficients for such cohomology groups, but only the weaker fact that the sequence

$$0 \to H_0^q(X) \otimes G \to H^q(X,G) \to H_0^{q+1}(X) * G \to 0 \tag{2}$$

is exact, i.e. that $H^q(X,G)/H_0^q(X) \otimes G \approx H_0^{q+1}(X) * G$. Here $H^q(X,G)$ is the q-dimensional spectral cohomology group of the space X with coefficient group G; $H_0^q(X) = H^q(X,I)$, where I is the additive group of integers; $\otimes$ is the tensor product sign; $*$ is the torsion product sign. Eilenberg and Mac Lane were able to correct the proof of formula (1), but only for metric compacta (([2]), the note on p. 820 and Theorem 44.2).

The universality of the group I in the general case was proved by me in the paper ([3]) (Chapter I, § 5), but without an explicit indication of the formula giving the group $H^q(X,G)$. The results of that paper were obtained during the 1940s (the paper was completely written by 1950), and therefore modern methods were not used in it; from their point of view the proof given there, after the introduction of the corresponding notions, becomes considerably shorter and more transparent. At the same time, as we shall now show, in the general case one can prove formula (1).

2. We introduce the following definitions. By a system of groups $\{G_\alpha; \pi\}$ we shall mean a collection of additive Abelian groups G_α, for some of which homomorphisms $\pi = \pi_\beta^\alpha$ of the group G_α into the group G_β are defined (in particular, if the indices α are directed partially ordered, then the system is called a direct, inverse, or two-sided spectrum when these homomorphisms are defined, respectively, for $\alpha < \beta$, $\alpha > \beta$, and in both cases). Two systems $\{G_\alpha; \pi\}$ and $\{H_\alpha; \omega\}$ are called conjugate if they are defined for the same set of indices and if the homomorphisms ω_α^β are defined if and only if the homomorphisms π_β^α are defined. The tensor product of two conjugate systems

$$\{G_\alpha; \pi\} \otimes \{H_\alpha; \omega\}$$

is called the factor group of the direct sum

$\sum_\alpha G_\alpha \otimes H_\alpha$ by the subgroup generated by all elements of the form $g_\alpha \otimes \omega_\alpha^\beta h_\beta - \pi_\beta^\alpha g_\alpha \otimes h_\beta$ $(g_\alpha \in G_\alpha,\ h_\beta \in H_\beta)$.

We have proved a theorem ((3), Chap. I, § 3, or (4)), which, with the aid of the definition introduced, can be written in the form

$$H^q(X, G) \simeq \{H_m^q(X);\ \pi_m^{m'},\ \omega_{m'}^m\} \otimes \{{}_mG;\ i_{m'}^m,\ j_m^{m'}\}, \tag{3}$$

where m runs through the nonnegative integers; $H_m^q(X) = H^q(X, I_m)$; $I_0 = I$; $I_m = I/mI$; mI is the subgroup of elements of the group I divisible by m (similarly, in general, mG and G_m are defined for an arbitrary group G); ${}_mG$ is the subgroup of elements $g \in G$ for which $mg = 0$; the homomorphisms $\pi_m^{m'}$, $\omega_{m'}^m$, $i_{m'}^m$, $j_m^{m'}$ are defined for $m \mid m'$; $i_{m'}^m$ is the embedding map of the subgroup ${}_mG$ into the subgroup ${}_{m'}G$; $j_m^{m'}$ is the map of ${}_{m'}G$ into ${}_mG$ arising as a result of multiplication of the elements of the group G by the number $\frac{m'}{m}$; $\pi_m^{m'}$ is a homomorphism of $H_{m'}^q(X)$ into $H_m^q(X)$, generated by reducing the coefficient group modulo m; $\omega_{m'}^m$ is a homomorphism of $H_m^q(X)$ into $H_{m'}^q(X)$, generated by the map I_m into $I_{m'}$ induced by multiplication by the number $\frac{m'}{m}$. Since the system $\{{}_mG;\ i_{m'}^m,\ j_m^{m'}\}$ is determined by specifying the group G, formula (3) shows that the system $\{H_m^q(X);\ \pi_m^{m'},\ \omega_{m'}^m\}$, which we call the **modular spectrum** of the cohomology groups of the space, is a universal system of cohomology groups, i.e. determines the cohomology groups of the space for any coefficient group.

3. We recall the proof of formula (3). Let $\sum u_i \otimes g_i$ be an element of the tensor product of the right-hand side of this formula $(u_i \in H_{m_i}^q(X),\ g_i \in {}_{m_i}G,\ i = 1, \dots, s)$. To it there corresponds uniquely an element (which we shall denote by $\sum g_i u_i$) of the group $H^q(X, G)$ containing the cocycle $l^q = \sum g_i l_i^q$, where l_i^q are integral cochains of some covering Ω^α of the space X which, after reduction modulo m_i, will be cochains from the cohomology classes u_i. Conversely, let $u \in H^b(X, G)$, and let l^q be a cocycle, contained in u, of some covering Ω^α of the space X. As is known, for every finite complex, and hence also for the nerve of the covering Ω^α, there exist such free generators l_i^q of the groups of integral cochains (the values assumed by

the index i for different values of the dimension q overlap only partially) that $\nabla l_i^q = m_{iq} l_i^{q+1}$, where ∇ is the coboundary operator, and m_{iq} are nonnegative integers (depending on i and q). Therefore $l^q = \sum g_i l_i^q$, $g_i \in G$, $\nabla l^q = m_{iq} g_i l_i^{q+1} = 0$ and, hence, $m_{iq} g_i = 0$, i.e. $g_i \in {}_{m_{iq}}G$, and consequently the element u always corresponds to some element $\sum u_i \otimes g_i$ of the tensor product from formula (3). Finally, if $\sum \bar{g}_k u_k = 0$, $\bar{g}_k \in {}_{\bar{m}_k}G$, $\bar{u}_k \in H^q_{\bar{m}_k}(X)$, then in some covering Ω^α we have $\sum \bar{g}_k \bar{l}_k^q \sim 0$, $\bar{l}_k^q \in \bar{u}_k$, i.e., decomposing with respect to the generators l_i^q of the above-mentioned basis (constructed for Ω^α),

$$\sum_{i,k} \bar{g}_k \lambda_{ik} l_i^q = \nabla \left(\sum_i g_i' l_i^{q-1} \right) = \sum_i m_i' g_i' l_i^q,$$

where $g_i' \in G$, whence $\sum_k \bar{g}_k \lambda_{ik} = m_i' g_i'$. On the other hand, the relation

$$\nabla \bar{l}_k^q = 0, \quad \text{i.e.} \quad \nabla \left(\sum_i \lambda_{ik} l_i^q \right) = \sum_i \lambda_{ik} m_i l_i^{q+1} \equiv 0 \pmod{\bar{m}_k}$$

gives $m_i \lambda_{ik} = 0 \pmod{\bar{m}_k}$, i.e. $\lambda_{ik} = \lambda_{ik}' \bar{m}_k / \delta_{ik}$, where λ_{ik}' are integers and $\delta_{ik} = (m_i, \bar{m}_k)$, and therefore

$$\sum_k \bar{u}_k \otimes \bar{g}_k = \sum_{i,k} \lambda_{ik}' \omega^{\frac{\delta_{ik}}{m_k}} \pi^{m_i}_{\delta_{ik}} u_i \otimes \bar{g}_k = \sum_{i,k} u_i \otimes \lambda_{ik}' \frac{\delta_{ik} \bar{m}_k}{m_i! \, \delta_{ik}!} \bar{g}_k = \sum_{i,k} u_i \otimes \lambda_{ik}' \frac{\bar{m}_k}{\delta_{ik}} \bar{g}_k = \sum_{i,k} u_i \otimes \lambda_{ik} \bar{g}_k = \sum_i u_i \otimes m_i' g_i' =$$

i.e. the correspondence between the left- and right-hand sides of formula (3) will be one-to-one (here $m_i = m_{i,q}$, $m_i' = m_{i,q-1}$, $m_i m_i' = 0$, u_i is a cohomology class containing the cochain l_i^q reduced modulo m_i).

4. Let us note that if G is a ring, then formula (3) makes it possible to define the cohomology ring of the space with coefficients in G from its cohomology rings mod m ($m = 0, 1, 2, ...$) and the mappings π and ω. For this, as is not hard to see, it is enough to define multiplication in the right-hand side of this formula as follows ($u_1 \in H^{q_1}_{m_1}(X)$, $u_2 \in H^{q_2}_{m_2}(X)$, $g_1 \in_{m_1} G$, $g_2 \in_{m_2} G$):

$$(u_1 \otimes g_1) \cdot (u_2 \otimes g_2) = (\pi^{m_1}_\delta u_1 \cdot \pi^{m_2}_\delta u_2) \otimes (g_1 g_2),$$

where $\delta = (m_1, m_2)$, which makes sense, since, evidently, $g_1 g_2 \in_\delta G$, i.e. $\delta g_1 g_2 = 0$ (for $\delta = \lambda_1 m_1 + \lambda_2 m_2$, where λ_1 and λ_2 are integers).

5. The universality of the group I for the spectral cohomology groups of a topological space can now be proved as follows. First we show without difficulty that the sequences

$$\cdots \to H^q_0(X) \xrightarrow{m} H^q_0(X) \xrightarrow{\pi} H^q_m(X) \xrightarrow{\chi} H^{q+1}_0(X) \xrightarrow{m} H^{q+1}_0(X) \to \cdots, \quad (4)$$

$$\cdots \to H_n^q(X) \xrightarrow{\omega} H_{mn}^q(X) \xrightarrow{\pi} H_m^q(X) \xrightarrow{\pi\chi} H_n^{q+1}(X) \xrightarrow{\omega} H_{mn}^{q+1}(X) \to \cdots, \tag{5}$$

where m denotes the endomorphism consisting in multiplication of all elements of the group by the number m, $\chi = \chi_0^m$ is the boundary homomorphism introduced by us in (9) (i.e. $\chi_0^m u$, where $u \in H_m^q(X)$, is the $(q+1)$-dimensional integral cohomology class containing the cocycle $\frac{1}{m}\nabla l$, where l, after reduction modulo m, belongs to u), and $\pi\chi$ is the superposition $\pi_n^0\chi_0^m$, are exact ((3), Ch. I, § 5, Lemma 2). Sequence (4), evidently, gives rise to a new exact sequence ($m \neq 0$)

$$0 \to [H_0^q(X)]_m \to H_m^q(X) \to {}_m[H_0^{q+1}(X)] \to 0, \tag{6}$$

where, as always, $[H_0^q(X)]_m = H_0^q(X)/mH_0^q(X)$.

Since all elements of the group ${}_m[H_0^{q+1}(X)]$ have orders bounded by the same number m, by Prüfer' s theorem ((5), p. 156), ${}_m[H_0^{q+1}(X)]$ decomposes into a direct sum of cyclic groups. We shall show that one can construct a mapping

$$\tilde{\chi} = \tilde{\chi}_0^m : \ {}_m[H_0^{q+1}(X)] \to H_m^q(X),$$

taking for each of the generating cyclic summands of the group ${}_m[H_0^{q+1}(X)]$ one of the elements of the group $H_m^q(X)$ which maps into it under the mapping χ. Then $\chi\tilde{\chi}$ is the identity mapping, and therefore the exact sequence (6) splits, i.e.

$$H_m^q(X) \simeq [H_0^q(X)]_m + {}_m[H_0^{q+1}(X)]. \tag{7}$$

The arbitrariness present in the construction of the mapping $\tilde{\chi}$ can be used in such a way as to obtain a kind of coherence of such direct decompositions. Namely, for each prime number p, first in the indicated way we construct a mapping

$$\tilde{\chi}_p^0 : \ {}_p[H_0^{q+1}(X)] \to H_p^q(X),$$

and then we carry out the further construction inductively: if $\tilde{\chi}_{p^{k-1}}^0$ has already been constructed, then for those generators a_i of the cyclic summands of the group ${}_{p^k}[H_0^{q+1}(X)]$ whose orders $\leq p^{k-1}$, we set

$$\tilde{\chi}_{p^k}^0 a_i = \omega_{p^k}^{p^{k-1}} \tilde{\chi}_{p^{k-1}}^0 a_i,$$

and for the generators of order p^k we take as $\tilde{\chi}_{p^k}^0 a_i$ one of the elements u of the group $H_{p^k}^q(X)$, for

for which $\pi_{p^{k-1}}^{p^k} u = \tilde{\chi}_{p^{k-1}}^0(pa_i)$, $\chi_0^{p^k} u = a_i$.} Since

$$\pi_0^{p^{k-1}} \chi_0^{p^k} \tilde{\chi}_{p^{k-1}}^0(pa_i) = \pi_0^0(pa_i) = 0,$$

then, by virtue of the exactness of sequence (5) for $m = p^{k-1}$, $n = p$, there exists an element u' satisfying the first of these conditions. Then

$$p(a_i - \chi_0^{p^k} u') = pa_i - \chi_0^{p^{k-1}} \pi_{p^{k-1}}^{p^k} u' = pa_i - \chi_0^{p^{k-1}} \tilde{\chi}_{p^{k-1}}^0 (pa_i) = pa_i - pa_i = 0,$$

and therefore, by the exactness of sequence (4), $a_i - \chi_0^{p^k} u' = \chi_0^p w$, and the element $u = u' + \omega_{p^k}^p w$ will also satisfy the second condition (here the obvious formulas are used $\chi_0^m = \chi_0^{m'} \omega_{m'}^m$, $\dfrac{m'}{m} \chi_0^{m'} = \chi_0^{m-m'} \pi_m^{m'}$ $(m \mid m')$).

For the mapping $\tilde{\chi}_{p^k}^0$ thus constructed we shall have $\chi_0^{p^k} \tilde{\chi}_{p^k}^0 a_i = a_i$, i.e. $\chi\tilde{\chi} = 1$. If $m = m_1 \cdots m_n$, where $m_i = p_i^{k_i}$ and $p_1, \dots, p_n$ are distinct primes, then we put

$$\tilde{\chi}_m^0 = \sum_i \omega_m^{m_i} \tilde{\chi}_{m_i}^0.$$

Then, as is easy to see, $\chi_0^m \tilde{\chi}_m^0 = 1$ will hold for all m.

Again using, in an analogous way, the exactness of sequence (5), we obtain without difficulty that

$$\pi_m^{m'} \chi_{m'}^0 a = \tilde{\chi}_m^0 \left(\frac{m'}{m} a \right), \qquad \omega_{m'}^{m''} \chi_{m'}^0 a = \tilde{\chi}_{m''}^0 a \, (m/m'/m'',\ a \in {}_m[H_0^{q+1}(X)]).$$

This defines the homomorphisms π and ω for the second summand of formula (7):

$$\pi_m^{m'} = j_m^{m'}, \qquad \omega_{m''}^{m'} = i_{m'}^{m''}.$$

For the first summand, however, which is a natural subgroup of the left-hand side of formula (7), $\pi_m^{m'}$ coincides with the embedding $\varphi_m^{m'}$ of the residue class mod m' into the residue class mod $m (m/m')$ containing it, and $\omega_{m''}^{m'}$ with $\psi_{m''}^{m'} (m'/m'')$—the passage from the residue class mod m' to the residue class mod m'' as a result of multiplication by the number m''/m'. Thus the modular spectrum of the cohomology groups of the space X is determined, and therefore formula (3) gives the group $H^q(X, G)$:

$$H^q(X, G) \approx ([H_0^q(X)]_m + {}_m[H_0^{q+1}(X)]; \pi, \omega) \otimes ({}_mG; i, j) \approx$$

$$\approx ([H_0^q(X)]_m; \varphi, \psi) \otimes ({}_mG; i, j) + ({}_m[H_0^{q+1}(X)]; j, i) \otimes ({}_mG; i, j).$$

6. It is easy to see that

$$([H_0^q(X)]_m; \varphi, \psi) \otimes ({}_mG; i, j) \quad (m = 0, 1, 2, \dots)$$

coincides with $H_0^q(X) \otimes G$. By virtue of the exactness of sequence (2) ((2), Theorem 40.3), it follows from this that

$$({}_m[H_0^{q+1}(X)]; j, i) \otimes ({}_mG; i, j) \approx H_0^{q+1}(X) * G,$$

which can also be proved directly from the definition of the torsion product (here already $m \neq 0$, i.e. $m = 1, 2, ...$). This gives formula (3).

Everything that has been said is also applicable to the relative cohomology groups of X mod A ($A \subset X$) and to the inner cohomology groups in the sense of P. S. Alexandrov (which take into account the noncompactness of the elements of a covering, ([6])). In an analogous way, formula (3) (with $q+1$ replaced by $q-1$) can be proved for singular homology groups of a space, which, however, was already known earlier, since it follows from general algebraic results (([7]), Ch. VI, Theorem 3.3). For singular cohomology groups and for spectral homology groups, formula (3) (and even the exact sequence (2)) does not hold in general, as is shown by counterexamples in ([8],[1]).

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Note: Figure translations are in progress. See original paper for figures.