# ON COMPUTATIONS CONNECTED WITH THE VERIFICATION OF THE RIEMANN HYPOTHESIS

1958

SovietRxiv

## Abstract

## Full Text

## MATHEMATICS

### N. A. MELLER

# ON COMPUTATIONS CONNECTED WITH THE VERIFICATION OF THE RIEMANN HYPOTHESIS

*(Presented by Academician A. A. Dorodnitsyn, 11 VI 1958)*

This note describes the results of computations carried out on the high-speed computer "Strela" at the Computing Center of the USSR Academy of Sciences and connected with the verification of the Riemann hypothesis concerning the zeros of the function

$$\zeta(s) = \sum_{\nu=1}^{\infty} \nu^{-s}.$$

As is known, the hypothesis consists in the assertion that all zeros of the indicated function (with the exception of the "trivial" zeros at the points $-2, -4, -6, \ldots$) lie on the line $\sigma = \operatorname{Re} s = 1/2$. In the present work the hypothesis was tested by the number of zeros from 15 000 to 35 337 and by $\tau$ from 2234 to 4735, where $s = \sigma + 2\pi i\tau$. In other words, all zeros from 15 000 to 35 337, if counted from the real axis, lie on the line $\sigma = 1/2$. (For computations devoted to verification of the hypothesis, see the works ($^{2,3}$).)

The method of carrying out the computations is as follows. The hypothesis was verified on separate intervals $(\tau_i, \tau_{i+1})$ of the line $\sigma = 1/2$. On the one hand, the number of zeros $N(\tau_i, \tau_{i+1})$ of the function $\zeta(s)$ inside the rectangle $0 < \sigma < 1$*, $\tau_i < \tau < \tau_{i+1}$, was counted; on the other hand, the number of changes of sign of the function

$$Z(\tau) = e^{i\theta}\zeta(1/2 + 2\pi i\tau), \tag{1}$$

where

$$e^{i\theta} = \frac{\pi^{-i\pi\tau}\Gamma(1/4 + i\pi\tau)}{|\Gamma(1/4 + i\pi\tau)|}$$

on the interval $\tau_i < \tau < \tau_{i+1}$, was counted.

The function $Z(\tau)$ is a real function of $\tau$, and its zeros coincide with the zeros of the Riemann function. Coincidence of the numbers of zeros counted by the two indicated methods confirms the hypothesis on the interval $(\tau_i, \tau_{i+1})$.

Let $N(\tau)$ be the number of zeros of the function $\zeta(s)$ in the rectangle $0 < \operatorname{Re} s < 1$, $0 < \operatorname{Im} s < 2\pi\tau$. Then

$$N(\tau_i, \tau_{i+1}) = N(\tau_{i+1}) - N(\tau_i). \tag{2}$$

For $N(\tau)$ the following formula is valid:

$$N(\tau) = \tau \ln \tau - \tau + \frac{7}{8} - \frac{1}{96\pi^2\tau} + \frac{1}{\pi}\Delta \arg \zeta(s) + R, \tag{3}$$

where $R = O\left(\frac{1}{t^3}\right)$, and $\Delta \arg \zeta(s)$ is the change of the argument of $\zeta(s)$ when $s$ varies along the broken line from 2 to $2 + 2\pi i\tau$ and then from $2 + 2\pi i\tau$ to $1/2 + 2\pi i\tau$. If $\tau$ is such that

$$\Delta \arg \zeta(s) < \frac{\pi}{2}, \tag{4}$$

$$\left|\left\{\tau \ln \tau - \tau + \frac{7}{8} - \frac{1}{96\pi^2\tau}\right\} - \frac{1}{2}\right| > R, \tag{5}$$

---

* It is known that for $\sigma \geq 1$ and for $\sigma \leq 0$ the function $\zeta(s)$ has no "nontrivial" zeros.

then

$$\begin{aligned} N(\tau) &= \left[\tau \ln \tau - \tau + \frac{7}{8} - \frac{1}{96\pi^2\tau}\right], && \text{if } \left\{\tau \ln \tau - \tau + \frac{7}{8} - \frac{2}{96\pi^2\tau}\right\} < \frac{1}{2}, \\ N(\tau) &= \left[\tau \ln \tau - \tau + \frac{7}{8} - \frac{1}{96\pi^2\tau}\right] + 1, && \text{if } \left\{\tau \ln \tau - \tau + \frac{7}{8} - \frac{1}{96\pi^2\tau}\right\} > \frac{1}{2}, \end{aligned} \tag{6}$$

where square brackets denote the integral part of a function, and braces the fractional part.

Condition (4) is always satisfied if $\operatorname{Re} \zeta(s) > 0$ along the broken line indicated above. Since $\operatorname{Re} \zeta(s) > 0$ for $\operatorname{Re} s = 2$, in order that condition (4) be satisfied it is enough that $\operatorname{Re} \zeta(s)$ be positive on the segment $[2 + 2\pi i\tau;\ 1/2 + 2\pi i\tau]$. Thus, in order that for a fixed $\tau$ one may use formula (6), it is necessary to check whether the following sum is positive for all $\sigma$ inside the interval $[0; 2]$:

$$\operatorname{Re}\zeta(\sigma+2\pi i\tau)=\sum_{\nu=1}^{[\sqrt{\tau}]}\frac{\cos 2\pi\tau\ln\nu}{\nu^{\sigma}}+\frac{\cos\left(\tau\ln\tau-\tau+\frac{1}{8}-\frac{1}{96\pi^2\tau}-\tau\ln\nu\right)2\pi}{\tau^{\sigma-1/2}\nu^{1-\sigma}}+R_1, \tag{7}$$

where $R_1=O(\tau^{-\sigma/2})$.

Divide the interval $[1/2;2]$ into $n$ parts. Let $\sigma_0=1/2,\sigma_1,\sigma_2,\dots,\sigma_n=2$ be the points of division of the interval. Denote by $\Sigma'$ the sum of all positive terms of expression (7), and by $\Sigma''$ the sum of all negative terms. Then for each of the intervals $[\sigma_k,\sigma_{k+1}]$ the following inequality holds:

$$\begin{aligned}\operatorname{Re}\zeta(\sigma+2\pi i\tau)>{}&\sum{}'\left(\frac{\cos 2\pi\tau\ln\nu}{\nu^{\sigma_k}}+\frac{\cos\left(\tau\ln\tau-\tau+\frac{1}{8}-\frac{1}{96\pi^2\tau}-\tau\ln\nu\right)2\pi}{\tau^{\sigma_k-1/2}\nu^{1-\sigma_k}}\right)\\&+\sum{}''\left(\frac{\cos 2\pi\tau\ln\nu}{\nu^{\sigma_{k+1}}}+\frac{\cos\left(\tau\ln\tau-\tau+\frac{1}{8}-\frac{1}{96\pi^2\tau}-\tau\ln\nu\right)2\pi}{\tau^{\sigma_{k+1}-1/2}\nu^{1-\sigma_{k+1}}}\right)+R_1,\end{aligned} \tag{8}$$

where $\sigma_k\le\sigma\le\sigma_{k+1}$.

Therefore, if for fixed $\tau$ it turns out that in formula (8)

$$\Sigma'+\Sigma''>|R_1| \tag{9}$$

for all intervals $[\sigma_k,\sigma_{k+1}]$, then $\operatorname{Re}\zeta(\sigma+2\pi i\tau)$ is positive for $1/2\le\sigma\le 2$, condition (4) is satisfied, and the number of zeros of the function $\zeta(s)$ inside the rectangle can be counted by formulas (6) and (2). In addition, condition (5) must also be satisfied.

The number $K_i$ of sign changes of the function $Z(\tau)$ inside the interval $(\tau_i,\tau_{i+1})$ was counted in the following way. The values of the function $Z(\tau)$ were computed successively at the points $\tau_i,\tau_i+\Delta\tau,\tau_i+2\Delta\tau,\dots,\tau_{i+1}$, where $\Delta\tau$ is a certain step, by the following formula:

$$\begin{aligned}Z(\tau)=2\sum_{\nu=1}^{[\sqrt{\tau}]}\nu^{-1/2}\cos\left(\frac{1}{2}\tau\ln\tau-\frac{\tau}{2}-\frac{1}{16}-\frac{1}{192\pi^2\tau}-\tau\ln\nu\right)2\pi+\\+(-1)^{[\sqrt{\tau}]-1}\tau^{-1/4}h(\{\sqrt{\tau}\})+R_2,\end{aligned} \tag{10}$$

where

$$h(\xi)=\frac{\cos 2\pi(\xi^2-\xi-1/16)}{\cos 2\pi\xi}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

and $R_2 = O(\tau^{-3/4})$. At the same time, each change of sign of the function was recorded.

The endpoints of the intervals $(\tau_i, \tau_{i+1})$ are chosen so that

$$|Z(\tau)| > R_2. \tag{11}$$

Consequently, in order that the point $\tau$ may serve as the endpoint of an interval within which the hypothesis is being checked, it is necessary that conditions (5), (9), and (11) be satisfied for it.

If the endpoints of the interval $(\tau_i, \tau_{i+1})$ are determined and if it turns out that $N(\tau_i, \tau_{i+1}) = K_i$, then in the indicated interval the hypothesis is true.

If, however, $N(\tau_i, \tau_{i+1}) > K_i$, then the number $K_i$ was counted again with step $\Delta\tau/2$, then, if necessary, with step $\Delta\tau/4$, $\Delta\tau/8, \ldots$. The step $\Delta\tau$ was chosen equal to 1/32. For most intervals such a step proved sufficient for verifying the hypothesis. In two cases the change of sign of the function $Z(\tau)$ could be detected only with step 1/1024. Finally, in one case (in the interval 2728.15; 2729.05) two zeros of the function were missed because of the insufficient accuracy of formula (10). Here it was necessary to use a more accurate formula:

**Fig. 1**

**Fig. 2**

$$Z(\tau) = e^{i\theta}\left(\sum_{\nu=1}^{n-1} \nu^{-1/2-2\pi i\tau} + \frac{1}{2} n^{-1/2+2\pi i\tau} + \frac{n^{1/2-2\pi i\tau}}{2\pi i\tau - \frac{1}{2}} + \sum_{\nu=1}^{k} T_\nu\right) + R_3, \tag{12}$$

where

$$T_\nu = \frac{B_{2\nu}}{(2\nu)!}\, n^{1/2-2\pi i\tau-2\nu} \prod_{j=0}^{2\nu-2}\left(\frac{1}{2} + 2\pi i\tau + j\right),$$

$B_{2\nu}$ is a Bernoulli coefficient.

For $R_3$ the following estimate is given:

$$|R_3| < \left(\frac{\tau}{n}\right)^{2k+1} \frac{4\pi\tau(2k)}{\sqrt{n}} \left(1+\frac{1}{2\pi\tau}\right)\left(1+\frac{2}{2\pi\tau}\right)\cdots\left(1+\frac{2k}{2\pi\tau}\right). \qquad (13)$$

Figures 1 and 2 give the graph of $Z(\tau)$ on the interval (2728.15; 2729.05) and the refined graph on the interval (2728.51; 2728.53).

Computing Center
of the Academy of Sciences of the USSR

Received
5 VI 1958

## CITED LITERATURE

1. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, IL, 1953.
2. A. M. Turing, Proc. London Math. Soc. (3), **3**, 99 (1953).
3. D. N. Lehmer, Acta Math., **95**, 3–4, 291 (1956).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*