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Abstract

Full Text

MATHEMATICS

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OSCILLATION THEOREMS FOR DIFFERENTIAL EQUATIONS OF HIGHER ORDERS AND THE SPECTRUM OF THE CORRESPONDING DIFFERENTIAL OPERATORS

(Presented by Academician S. N. Bernstein, 15 VII 1957)

Between the oscillatory properties of the solutions of the equation

$$l_2[y] \equiv -y'' + q(x)y = \lambda y \qquad (0 \leq x \leq \infty) \tag{1}$$

and the spectrum of any self-adjoint operator L generated by the operation l_2, there is a well-known connection, by virtue of which the set of points of the spectrum preceding the point $\lambda = \lambda_0$ will be finite or infinite depending on whether, for $\lambda = \lambda_0$, equation (1) is nonoscillatory or oscillatory (i.e., whether each of its solutions has a finite or an infinite number of zeros).

The indicated connection is usually used in the investigation of the spectrum; namely, starting from criteria of nonoscillation or oscillation, one establishes certain properties of the spectrum. A classical example of this kind is H. Weyl' s proof of the criterion for discreteness of the spectrum (([1], p. 73). Another proof of this criterion was given by the author in ([2a]) with the aid of the decomposition method introduced in that paper.

In the present note the usual course of investigation is reversed; namely, with the aid of decomposition the spectrum is studied directly, and from this conclusions are drawn about the oscillation of the differential equation. In doing so, there is no need to use any asymptotic properties of the solutions of the differential equation.

Such a path leads to a natural formulation of oscillation problems for differential equations of higher orders of the form

$$l[y] \equiv \sum_{k=0}^{n} (-1)^{n-k} \left[p_k(x) y^{(n-k)}\right]^{(n-k)} = \lambda y \qquad (p_0(x) = 1,\ 0 \leq x < \infty) \tag{2}$$

or

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$$(-1)^n y^{(2n)} + \ddot{q}(x)y = \lambda y \qquad (0 \le x < \infty). \tag{3}$$

At the basis of what follows lies the following lemma, easily established with the aid of decomposition.

Lemma 1. Let $\widetilde{L}$ be some self-adjoint operator generated by the operation l, and let U be the negative part of the spectrum of the operator $\widetilde{L}$. In order that the set U be bounded below and discrete, it is necessary and sufficient that for every $\varepsilon > 0$ there exist α such that the quadratic functional

$$\Phi_\varepsilon[y] = \int_\alpha^\infty l[y]\overline{y}\,dx + \varepsilon \int_\alpha^\infty |y|^2\,dx \tag{4}$$

would be nonnegative. In order that the set U be finite, it is necessary and sufficient that, for some α, the functional $\Phi_0[y]$ be nonnegative.

In all cases, the admissible functions for the functional $\Phi_\varepsilon[y]$ are taken to be any finite functions from $D_{\tilde{L}}$ that are equal to zero near α.

The definition of oscillation adopted below (for $n = 2$, see [(3)]) is such that, in passing from equation (1) to equation (2), the connection with the spectral properties mentioned at the beginning of the note is preserved.

Definition. Equation (2) is called **oscillatory** if, for every α, there exists a solution of this equation having more than one n-fold zero to the right of α. Otherwise equation (2) is called nonoscillatory.

By means of the splitting method, Theorem 1 is easily established.

Theorem 1. *In order that equation (2) be nonoscillatory for $\lambda = \lambda_0$, it is necessary and sufficient that the part of the spectrum of the operator $\tilde{L}$ lying to the left of the point $\lambda = \lambda_0$ be an infinite set.*

It can be shown that, in the case of oscillation of equation (2), the first of the n-fold zeros of the solution mentioned in the definition may be prescribed arbitrarily.

The negative part of any function $f(x)$ will be denoted below by $f^*(x)$, so that $f^*(x) = \min\{0, f(x)\}$.

Theorem 2. *If, for every $\delta > 0$, the inequality*

$$\int_{M_{k\delta}} |p_k^*(x)|\,dx < \infty \qquad (k = 1, 2, \dots, n), \tag{5}$$

holds, where $M_{k\delta}$ is the set of values x for which $|p_k^*(x)| \ge \delta$, then equation (2) is nonoscillatory for $\lambda < 0$ (i.e. the negative part of the spectrum of the operator $\tilde{L}$ is bounded below and discrete).

From this, in particular, follows the result of I. M. Rapoport (4), who established, by means of asymptotic formulas for the solutions of equation (2), the validity of Theorem 2 under the assumption of summability of all coefficients $p_k(x)$ on the half-axis $x > 0$.

From Theorem 2 there also follows the nonoscillation of equation (2) for $\lambda < 0$ under the inequalities

$$\int_0^\infty |p_k^*(x)|^{r_k}\,dx < \infty,$$

where $r_k \geq 1$ $(k = 1, 2, \dots, n)$.

In the classical case $n = 1$, the nonoscillation of equation (1) for $\lambda = 0$, as is known, is equivalent to the existence of a solution of the corresponding Riccati equation on some half-axis $[\alpha, \infty)$. In the general case, the nonoscillation of equation (2) for $\lambda = 0$ is equivalent to the existence of a solution of a certain nonlinear system of differential equations on the interval $[a, b)$ for some a and any $b > a$.

To construct this system it is sufficient to use Lemma 1 and the theorem of M. G. Krein (5) on representability, in the case of nonnegativity of the functional $\Phi_0[y]$, of the operation l in the form

$$l = \mu' \mu, \tag{6}$$

where

$$\mu[y] = y^{(n)} + u_1(x)y^{(n-1)} + \dots + u_n(x)y, \qquad \mu'[y] = (-1)^n y^{(n)} +$$
$$+ (-1)^{n-1}[u_1(x)y]^{(n-1)} + \dots + u_n(x)y.$$

Equating the coefficients of the derivatives $y^{(k)}$ $(k = 0, 1, \dots, 2n-1)$ in both sides of equality (6), we obtain the required system of differential-

equations with respect to the functions $u_k(x)$ $(k = 1, 2, \dots, n)$, which for $n = 1$ reduces to a single Riccati equation

$$u'^2 - u^2 - q(x) = 0. \tag{7}$$

In 1948 N. Adamov (^{6}), studying equation (7), established* the convexity of the set of functions $q(x)$ for which this equation has a solution on some half-axis $[\alpha, \infty)$ (i.e., for which equation (1) is nonoscillatory). A generalization of this fact to equation (2) and to the nonlinear system of differential equations associated with it follows directly from Lemma 1. From the same lemma it follows that, if equation (2) is nonoscillatory, then an equation with larger coefficients is also nonoscillatory.

In the particular case of equation (2) with constant coefficients $p_k(x) = a_k$, the set K_a of points $Q(a_1, a_2, \dots, a_n)$ of the n-dimensional coefficient space corresponding to equations (2) that are nonoscillatory for $\lambda = 0$ is the closure of the

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set of points for which, in the sequence of the $2n$ first principal minors of the Hankel matrix $\|s_{j+k}\|_{j,k=0,1}^{n}$, where $k\alpha_k = s_1\alpha_{k-1} - s_2\alpha_{k-2} + s_3\alpha_{k-3} - \cdots \pm s_k$, $\alpha_{2k} = a_k,\ \alpha_{2k+1} = 0$ $(k = 0, 1, \dots, n)$, there are n changes of sign.

By means of the change of variables

$$x = \ln t, \qquad y = x^{\frac{1-2n}{2}} z \tag{8}$$

the functional $\Phi_0[y]$ corresponding to equation (2) with constant coefficients is reduced to the form

$$\widetilde{\Phi}_0[z] = \int_{\alpha'}^{\infty} \left|z_t^{(n)}\right|^2 dt + \sum_{k=1}^{n} \int_{\alpha'}^{\infty} b_k t^{-2k} \left|z_t^{(n-k)}\right|^2 dt,$$

where the numbers b_k are linear functions of the coefficients a_k,

$$b_k = \varphi_k(a_1, a_2, \dots, a_k) \qquad (k = 1, 2, \dots, n), \tag{9}$$

whence Theorem 3 follows.

Theorem 3. *Let the convex set K_b be the image of the set K_a determined by the transformation (9), and let*

$$b'_k = \liminf_{x\to\infty} p_k(x), \qquad b''_k = \limsup_{x\to\infty} p_k(x) \qquad (k = 1, 2, \dots, n).$$

If $Q(b'_1, b'_2, \dots, b'_n) \in K_b$, then equation (2) for $\lambda = 0$ is nonoscillatory. If $Q(b''_1, b''_2, \dots, b''_n) \notin K_b$, then equation (2) for $\lambda = 0$ is oscillatory.

In particular, for $n = 1$ the set K_b is the half-axis $b_1 \geq -\frac{1}{4}$ (Kneser); for $n = 2$ the set K_b is determined by the inequalities: $b_2 \geq -\frac{9}{4}b_1 - \frac{9}{16}$ for $b_1 \geq -\frac{5}{2}$; $b_2 \geq \frac{1}{4}(2 - b_1)^2$ for $b_1 \leq -\frac{5}{2}$ (3). For $n = 3$ the set K_b is a part of the space containing the first octant and bounded by the surface $b_1 = \varphi_1(u - 2v)$, $b_2 = \varphi_2(u - 2v,\ v^2 - 2uv)$, $b_3 = \varphi_3(u - 2v,\ v^2 - 2uv,\ uv^2)$, where $u \geq 0,\ v \geq 0$.

By iterating the transformation (8), Theorem 3 admits a development in the direction indicated for $n = 1$ by Hille (7).

In the particular case of a two-term operation, Theorem 4 holds.

Theorem 4. *Equation (3) is nonoscillatory if*

$$q(x) \geq -\alpha_n^2 x^{-2n},$$

and is oscillatory if, for some $\delta > 0$,

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$$q(x) < -(\alpha_n^2 + \delta)x^{-2n},$$

* Under the assumption of periodicity of $q(x)$.

where the "Kneser constant" α_n^2 is determined by the formula $\alpha_n = \dfrac{(2n-1)!!}{2^n}$.

The first part of this theorem is refined by Theorem 5.

Theorem 5. If for every $\eta > 0$ the inequality

$$\int_{M_\eta} x^{2n-1}|q^*(x)|\,dx < \infty,$$

where M_η is the set of values of x for which $x^{2n}|q^*(x)| \geq \alpha_n^2 - \delta$, holds, then equation (3) for $\lambda = 0$ is nonoscillatory.

The conditions of Theorem 5 are, in particular, satisfied if for some $r \geq 1$

$$\int_0^\infty x^{2nr-1}|q^*(x)|^r\,dx < \infty.$$

For $n = 1$, $r = 1$, this yields the well-known criterion for nonoscillation of solutions of equation (1) for $\lambda = 0$ (^{8}).

Further conditions for oscillation are obtained by means of the method used in the author' s note (^{26}).

Theorem 6. If the function $q(x)$ satisfies the condition

$$\int_0^\infty q(x)\,dx = -\infty,$$

then equation (3) for $\lambda = 0$ is oscillatory (for $n = 1$, under the assumption $q(x) \leq 0$, see (^{8})).

Theorem 7. If $q(x) \leq 0$ for large x and

$$\liminf_{\rho\to\infty} \rho^{2n-1} \int_\rho^\infty |q(x)|\,dx > A_n^2,$$

where

$$A_{n+1} = (2n+1)^{-1/2} \left(\sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}}{2n-k+1} \right)^{-1} n!,$$

SovietRxiv

then equation (3) for $\lambda = 0$ is oscillatory.

The second part of Theorem 4 is refined by Theorem 8.

Theorem 8. If $q(x) + \alpha_n^2 x^{-2n} \leq 0$ for large x and

$$\liminf_{\rho\to\infty} \ln\rho \int_\rho^\infty x^{2n-1} \left|q(x) + \alpha_n^2 x^{-2n}\right| \, dx > B_n^2,$$

where

$$B_n^2 = \frac{n(4n^2-1)}{3\cdot 4^{n-1}} \sum_{k=1}^{n} \frac{1}{2k-1} \sum_{k=0}^{2n-2} \frac{(-1)^k \binom{2n-2}{k}}{4n-3-k} \left[\sum_{k=1}^{n} \frac{(-1)^{k-1}\binom{n-1}{k-1}}{2n-k}\right]^{-2},$$

then equation (3) for $\lambda = 0$ is oscillatory.

In the last theorems one may replace $\liminf_{\rho\to\infty}$ by $\lim_{\rho_k\to\infty}$.

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