Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1958. Volume 123, No. 6

MATHEMATICS

O. M. KOZLOV

FRACTIONAL POWERS OF SELF-ADJOINT EXTENSIONS OF OPERATORS AND SOME BOUNDARY-VALUE PROBLEMS

(Presented by Academician I. G. Petrovsky, 4 VIII 1958)

1. Let, in a bounded simply connected domain G of n-dimensional space with sufficiently smooth boundary Γ, a self-adjoint differential operator L be given.

We shall study the parabolic-type equation

$$\frac{\partial u}{\partial t} + Lu = 0 \tag{1}$$

under homogeneous self-adjoint boundary conditions whose coefficients depend on time t.

In the space $L_2(G)$ of square-summable functions, such a problem corresponds to an equation of the form

$$\frac{du}{dt} + A(t)u = 0, \tag{2}$$

where $A(t)$ is a certain self-adjoint operator in $L_2(G)$, generated by the operator L and the given boundary conditions. Moreover, if the coefficients of L do not depend on t, then $A(t)$ for different t may be regarded as different self-adjoint extensions of one and the same symmetric operator A with finite (in the case $n = 1$) or infinite deficiency index.

Equation (2) has recently been studied in many papers. In the case under consideration, the presence of variable boundary conditions introduces the difficulty that the domain of definition of the operator $A(t)$ changes with time. On the other hand, it is known ([1]) that, for example, for a self-adjoint elliptic operator of second order with boundary condition

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma} - \sigma(t,s)|_{\Gamma} = 0 \tag{3}$$

the domain of definition of the square root of the operator does not depend on t. For differential operators of higher order, there are possible such boundary conditions under which only the operators $A^{\gamma}(t)$ with $\gamma < \frac{1}{2}$ have a common domain of definition. For example, for an operator of the 4th order with one "natural" (see ([1])) and one condition of the form (3), the roots of the 4th degree of the operator have a common domain of definition. In connection with this there arises the problem of studying fractional powers of self-adjoint extensions depending on t. For ordinary differential operators some results were obtained in ([2]). We shall present one of these results in a somewhat different form.

Theorem 1. Let $A(t)$ be a self-adjoint positive-definite operator generated by the ordinary $(n = 1)$ regular differential operator

$$Lu(x) = u^{[2m]}(x) = p_m(x)u(x) - \frac{d}{dx}\left[p_{m-1}(x)\frac{du}{dx} - \frac{d}{dx}\left[p_{m-2}(x)\frac{d^2u}{dx^2} - \cdots\right.\right.$$

$$\left.\left.\cdots - \frac{d}{dx}\left[p_1(x)\frac{d^{m-1}u}{dx^{m-1}} - \frac{d}{dx}\left[p_0\frac{d^m u}{dx^m}\right]\right]\cdots\right]\right] \tag{4}$$

and by the boundary conditions

$$\sum_{k=0}^{2m-1} \alpha_{jk}u^{[k]}(a) + \beta_{jk}u^{[k]}(b) = 0 \qquad (j = 0, \dots, 2m-1). \tag{5}$$

If the coefficients in the principal* boundary conditions do not depend on t, and the coefficients in the natural conditions satisfy a Lipschitz condition of order α in t, $\mathrm{Lip}\,\alpha$ $(0 < \alpha \leqslant 1)$, then for $0 < \gamma < 1/2$ the inequality

$$\|[A^{\gamma}(t) - A^{\gamma}(\tau)]\, A^{-\gamma}(\tau)\| \leqslant C|t-\tau|^{\alpha} \tag{6}$$

holds.

P. E. Sobolevskii ([4]) showed that the fulfillment of condition (6) for some $\gamma < 1$ and $\alpha > 1 - \gamma$ is sufficient for the existence of a solution of equation (2).

Combining Theorem 1 with the result of P. E. Sobolevskii gives:

Theorem 2. Let, for the operator L given by formula (4) and the boundary conditions (5), the assumptions of Theorem 1 be satisfied for some $\alpha > 1/2$.

Then for any function $\varphi(x) \in L_2(a,b)$ there exists a solution $u(t,x)$ $(0 < t < \infty)$ of equation (1), satisfying the boundary conditions (5) and the initial condition: as $t \to 0$ in the mean-square sense,** $u(t,x) \to \varphi(x)$.

In the case of operators with partial derivatives, condition (6) has not been verified directly. However, Sobolevskii' s proof of the theorem mentioned above ([4], Theorem 1) turns out to remain valid if, for $\gamma = 1/2$, condition (6) is replaced by the following:

$$\left\|A^{1/2}(t)A^{-1/2}(\tau) - A^{-1/2}(t)A^{1/2}(\tau)\right\| \leqslant C|t-\tau|^{\alpha}, \tag{7}$$

where $\alpha > 1/2$.

Using methods of the theory of extensions of operators ([3],[5],[6]), this condition can be verified for elliptic operators of the second order.

Theorem 3. Let

$$Lu = -\sum_{i,k=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ik}(x)\frac{\partial u}{\partial x_k}\right) + c(x)u, \tag{8}$$

where $c(x) \geqslant 0$ and the eigenvalues of the matrix $\{a_{ik}(x)\}$ are bounded below by a positive number m. Suppose that there exist first derivatives of the functions $a_{ik}(x)$, continuous in the closed domain $G+\Gamma$, and that the function $c(x)$ satisfies in $G+\Gamma$ a Lipschitz condition of order β $(0 < \beta < 1)$.

If $\sigma(t,s)$ is bounded for each t and measurable in s, $\sigma(t,s) \geqslant \sigma(s) \geqslant 0$,*** where $\int_\Gamma \sigma(s)\,ds \neq 0$, then for the self-adjoint operators $A(t)$ defined by

* A boundary condition of the form (5) is called principal if $\alpha_{jk} = \beta_{jk} = 0$ for $k \geqslant m$ ([3]). All non-principal conditions are called natural if from their totality it is impossible, by means of linear combinations, to obtain a principal condition. The system (5) can always be replaced by an equivalent system in which all non-principal conditions are natural.

** The derivative with respect to t in equation (1) is also understood as a derivative in the mean-square sense.

*** If $c(x) \not\equiv 0$, then it is sufficient that $\sigma(t,s) \geqslant 0$.

by the differential expression (8) and boundary condition (3), the inequality holds

$$\left\|A^{1/2}(t)A^{-1/2}(\tau) - A^{-1/2}(t)A^{1/2}(\tau)\right\| \leq C\Delta\sigma,$$

where

$$\Delta\sigma = \operatorname{vrai}\sup_{s\in\Gamma} |\sigma(t,s) - \sigma(\tau,s)|.$$

Subsequently, an analogous assertion for a more general case was obtained by P. E. Sobolevskii by another method.

Apparently, in general it is easier to verify a condition of type (7)

$$\|A^{\gamma}(t)A^{-\gamma}(\tau) - A^{-\gamma}(t)A^{\gamma}(\tau)\| \leq C|t-\tau|^{\alpha}, \tag{9}$$

than condition (6). Obviously, (9) follows from (6), but not conversely. However, the following holds:

Theorem 4. *Let, in some Hilbert space, the self-adjoint operators $S(t)$ have a common domain of definition and satisfy the condition*

$$\left\|S(t)S^{-1}(\tau) - S^{-1}(t)S(\tau)\right\| < C|t-\tau|^{\alpha}.$$

Then for any positive $\beta < 1$

$$\left\|[S^{\beta}(t) - S^{\beta}(\tau)]S^{-\beta}(\tau)\right\| \leq C_1(\beta)|t-\tau|^{\alpha}.$$

The last assertion makes it possible to formulate Sobolevskii' s theorem in a form more convenient for applications.

Theorem 5. *Let a self-adjoint operator $A(t)$, $0 \leq t \leq T$, be given in a Hilbert space H, such that $(A(t)u,u) \geq m(u,u)$. Let, for some $\rho < 1$, the operators $A^{\rho}(t)$ have a common domain of definition and satisfy the condition*

$$\|A^{\rho}(t)A^{-\rho}(\tau) - A^{-\rho}(t)A^{\rho}(\tau)\| \leq C|t-\tau|^{1-\rho+\varepsilon} \qquad (\varepsilon > 0).$$

Then for every $u_0 \in H$, for $t > 0$ there exists a solution of equation (2), satisfying the initial condition $\lim_{t\to 0} u(t) = u_0$. Moreover

$$u(t) \in D(A^{1+\varepsilon'}), \qquad (0 < t \leq T).$$

2. In this section we shall present some auxiliary results obtained in the study of the operators $A(t)$ as various self-adjoint positive-definite extensions of one and the same symmetric operator A, for which $D(A^{1/2}(t))$ does not depend on t.

Following M. I. Vishik ([5]) and M. Sh. Birman ([6]), we associate with the operator $A(t)$ the operator $B(t) = A^{-1}(t) - A_{\mu}^{-1}$, where A_{μ} is the hard, in the terminology of M. G. Krein ([3]), self-adjoint extension of the operator A. The operator $B(t)$ is a bounded self-adjoint and, as shown in ([3]), positive operator, mapping the whole space H into the subspace U, consisting of solutions in H of the equation $Lu = 0$. The operators $B^{1/2}(t)$ have a common range, whose closure we denote by V. In V the operator $B^{-1/2}(t)$ is naturally defined.

With the aid of results of M. Sh. Birman ([6]), one establishes:

Lemma 1. *The norm of the operator $A^{1/2}(t)A^{-1/2}(\tau) - A^{-1/2}(t)A^{1/2}(\tau)$ in H is equal to the norm of the operator $B^{1/2}(t)B^{-1/2}(\tau) - B^{-1/2}(t)B^{1/2}(\tau)$.*

For the case where $A(t)$ is defined by formula (8) and boundary condition (3), the following is true:

Lemma 2. For the operator $B(t)$ in U the formula holds

$$B(t) = KQ^{1/2}[P + \sigma(t)]^{-1}Q^{1/2}K^{-1}, \tag{10}$$

where K, P, Q are operators independent of t; K is a certain isometric operator mapping $L_2(\Gamma)$ onto U; Q is a positive bounded self-adjoint operator in $L_2(\Gamma)$, defined by the formula
$Q\varphi = -\dfrac{\partial}{\partial \nu} A_\mu^{-1} u\Big|_\Gamma$, where $Lu = 0$, $u|_\Gamma = \varphi$; P is the positive-definite operator introduced by M. I. Vishik ([5]).*

The representation (10) of the operator B differs from the representations in (5) and (7) in that, apart from the isometries K and K^{-1}, it involves only bounded operators acting in the same Hilbert space $L_2(\Gamma)$, which facilitates estimates.

The operators $(P + \sigma I)^{-1}$ and Q, on the subspace of functions orthogonal to the unit function (and when $c \neq 0$ or else $\int_\Gamma \sigma \, dS \neq 0$, on all of $L_2(\Gamma)$), are comparable in the sense that

$$m(Q\varphi, \varphi) \leq ([P + \sigma I]^{-1}\varphi, \varphi) \leq M(Q\varphi, \varphi),$$

where m and M are some positive constants. The last inequality can be expressed in classical terms.

Theorem 6. Let $\varphi(s) \in L_2(\Gamma)$; let $u(x)$ be the solution of the Dirichlet problem $L_1 u = 0$, $u|_\Gamma = \varphi$; let $v(x)$ be the solution of the problem
$L_2 v = 0$, $\dfrac{\partial v}{\partial \nu} + \sigma v\Big|_\Gamma = \varphi$, where L_1 and L_2 are operators of the form (8).

Then the inequalities

$$m\|u\|_{L_2(G)} \leq \|v\|_{W_2^{(1)}(G)} \leq M\|u\|_{L_2(G)}$$

hold.

In the proof of Theorem 6 the results of M. I. Vishik ([5]) and P. Lax ([8]) are used.

The present work was carried out under the supervision of S. G. Krein, to whom the author expresses his deep gratitude.

Voronezh
Forestry Engineering Institute

Received
2 VIII 1958

REFERENCES

1. S. G. Mikhlin, *The Minimum Problem for a Quadratic Functional*, Moscow–Leningrad, 1952.

2. O. M. Kozlov, Trudy seminara po funktsional’ n. analizu, Voronezh State Univ., No. 6 (1958).

3. M. G. Krein, Mat. sbornik, 20 (62), No. 3 (1947).

4. P. E. Sobolevskii, DAN, 123, No. 6 (1958).

5. M. I. Vishik, Trudy Moskov. matem. obshch., 1 (1952).

6. M. Sh. Birman, Mat. sbornik, 38 (80), No. 4 (1956).

7. M. Sh. Birman, DAN, 92, No. 2 (1953).

8. P. D. Lax, Comm. Pure and Appl. Math., 10, No. 4 (1957).

* The operator P is defined as the Friedrichs self-adjoint extension of the operator originally given by the formula

$$P\varphi = \frac{\partial u}{\partial \nu}\Big|_{\Gamma}$$

on the set of boundary values of harmonic functions belonging to $W_2^2 G$.

Note: Figure translations are in progress. See original paper for figures.