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Abstract

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ON THE QUESTION OF THE CONNECTION BETWEEN PROXIMITY SPACES AND BICOMPACT EXTENSIONS OF COMPLETELY REGULAR SPACES

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1. Yu. M. Smirnov showed ([1]) the existence of a natural one-to-one correspondence between all bicompact extensions of a given completely regular space R and all proximity spaces having R as their carrier and compatible with the topology given in R. The purpose of the present note is to obtain this important result from the general theory of commutative normed rings.

Following ([1]), we formulate certain definitions needed below.

A set P, in which certain subsets A and B are declared to be close, is called a **general proximity space** if the following axioms are satisfied:

B1. If A is close to B, then B is close to A.

B2. The sum $A \cup B$ is close to C if and only if at least one of the sets A or B is close to C.

B3. Every element of the set P is close to itself.

B4. All of P is far from the empty set.

A general proximity space is called simply a **proximity space** if, in addition to axioms B1–B4, the following separation axioms are also satisfied:

T1. If points p and q of P are close to each other, then they coincide.

B$_\rho$. If A and B are far from each other, then there exists a decomposition of the space P into two parts D_1 and D_2 such that $A \subset D_1$, $B \subset D_2$, A is far from D_2, and B is far from D_1.

Closeness of sets is denoted by the symbol Δ (for example, $A\Delta B$), and absence of closeness by the symbol $\overline{\Delta}$.

From axioms B2 and B3 it follows at once that any two intersecting sets are close to each other. However, from the fact that the sets A and B do not intersect, it does not yet follow that $A\overline{\Delta}B$.

2. Let there be given a certain general proximity space R, which at the same time is also a topological space. We shall say that the **closeness given in R is compatible with the topology present in this space if any two sets whose closures intersect are close to each other**.*

In a topological space one can, generally speaking, introduce many different proximities compatible with the topology present in it and not equivalent to one another; these proximities $\{\Delta_\alpha\}$ form a partially ordered–

* In ([1]) compatibility (there the term "compatibility" is used) of proximity and topology is defined somewhat differently. In the case of proximity spaces the equivalence of these definitions is easy to verify.

system if we agree that $\Delta_1 < \Delta_2$ if and only if $A\Delta_2 B$ implies $A\Delta_1 B$.

Let P be a general proximity space. A bounded numerical function $f(x)$ defined on P is called Δ-continuous if from the fact that $A\Delta B$ it follows that the set $\overline{f(A)}\cap\overline{f(B)}$ is nonempty, i.e., that f does not separate the sets A and B. We shall need the following result, established essentially already by B. A. Efremovich ([2]), in a somewhat different form.

Theorem 1. *If the proximity Δ, defined in a topological space R, is compatible with the topology present in R, then every Δ-continuous function on R is continuous.*

The basis for applying the theory of normed rings to the study of proximity spaces is the following theorem:

Theorem 2. *The collection of all Δ-continuous functions defined on any general proximity space forms a ring, closed with respect to uniform convergence and containing, together with each function f, the function complex-conjugate to it.*

Proof. Let the functions f_1 and f_2 be Δ-continuous. We shall show that their sum $f = f_1 + f_2$ is also Δ-continuous. Let the sets A and B be such that

$$\overline{f(A)} \cap \overline{f(B)} = 0. \tag{1}$$

We shall show that in this case $\overline{A}\Delta B$. The function f is bounded; therefore it follows from (1) that the distance between the sets $f(A)$ and $f(B)$ exceeds some positive quantity ε. Partition the sets A and B into pairwise disjoint subsets:

$$A = \bigcup_{i=1}^{n} A_i, \qquad B = \bigcup_{k=1}^{m} B_k$$

so that on each of these subsets the oscillations of the functions f_1 and f_2 are less than $\varepsilon/4$. Then, from (1) and the choice of ε, it follows that every pair of sets A_i and B_k is separated by at least one of the functions f_1, f_2. But then $A_i\overline{\Delta}B_k$; by axiom B_2 it follows that $A\overline{\Delta}B$.

Further, it is obvious that if the function f is Δ-continuous, then so is kf, where $k = \text{const}$, and f^2 is also Δ-continuous. Hence, and from the proved Δ-continuity of the sum, in view of the equality

$$fg = \tfrac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$$

we obtain that the product of Δ-continuous functions is Δ-continuous.

Thus, the Δ-continuous functions on R form a ring. Let us show that it is closed in the sense of uniform convergence. Let

$$f(x) = \lim_{n\to\infty} f_n(x)$$

and all the functions $f_n(x)$ be Δ-continuous. If $f(x)$ separates some two sets A and B, then, evidently, all $f_n(x)$ also separate them for sufficiently large n. Consequently, $A\overline{\Delta}B$. This establishes the Δ-continuity of $f(x)$. The Δ-continuity of the function complex-conjugate to a Δ-continuous one is obvious.

3. Let R be some topological space and C an arbitrary subring of the ring of bounded continuous functions on it. Define in R the proximity of sets by setting that $A\overline{\Delta}B$ if and only if there exists in C a function $f(x)$ separating these two sets. It is easy to verify that the proximity thus defined in R satisfies axioms B1–B4, as well as axiom B_5. If, however, the subring C is such that for two points $x, y \in R$ there exists in C a function separating them,

(for such subrings to exist it is necessary to assume that R is completely regular), then the corresponding general proximity space also satisfies axiom T_1, i.e. is a proximity space. The proximity that is defined in the topological space R in the manner indicated above, with the aid of the subring C of the ring of all continuous functions on R, is called a **functional proximity** and is denoted by Δ_C.

Theorem 3. Let R be a topological space, Δ a certain general proximity in R, compatible with the topology defined in R, and C the ring of all Δ-continuous functions on R corresponding to Δ. Then $\Delta_C < \Delta$. If, however, R is a completely regular space and the proximity Δ satisfies the axioms T_1 and B_ρ, then $\Delta = \Delta_C$.

Proof. The first assertion of the theorem follows directly from the definition of Δ-continuity. Indeed, if $A\Delta B$, then A and B are not separated by any function from C, and consequently $A\Delta_C B$.

To prove the second assertion of the theorem it is enough to show that in this case, for any two remote sets A and B, there is a Δ-continuous function separating them. The existence of such a function for any two remote sets was proved in the work of V. A. Efremovich ([2]).

4. As is known ([3]), every closed subring C of the ring of all bounded continuous functions on a completely regular space R such that for any two points of R there is in C a function separating them generates a certain bicompact extension R_C of the space R (this extension is constructed as the space of maximal ideals of the ring C). Conversely, every bicompact extension of the space R can be obtained in the indicated way with the aid of some subring C. The correspondence thus obtained between all subrings of the ring of bounded continuous functions on R that separate any two points of R, and all bicompact extensions of this space, preserves order in the sense that if $C_1 \subseteq C_2$, then R_{C_1} is a continuous image of R_{C_2}.

The correspondence between subrings C and all proximities compatible with the topology defined on the completely regular space R, which is established by Theorem 3, possesses an analogous order-preserving property. From the comparison of these two facts there follows the correspondence, discovered by Yu. M. Smirnov, between proximity spaces and bicompact extensions. In this case two sets A and B are close in the sense of the proximity determined by the subring C if and only if their closures in the bicompact extension corresponding to this subring have a nonempty intersection.

The topology of a completely regular space R admits a unique proximity compatible with it if and only if the ring of all bounded continuous functions on R has no proper subrings separating any two points of R. This is certainly the case if R is bicompact, but not only in this case.

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Note: Figure translations are in progress. See original paper for figures.