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# Mathematics

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**Abstract**

**Full Text**

**Mathematics**

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**ON THE REPRESENTATION OF LINEAR FUNCTIONALS IN THE CLASS OF DISCONTINUOUS FUNCTIONS**

As is known, every linear functional  $A[f]$ , defined in the class  $C_0$  of continuous functions given on the interval  $(a, b)$ , is represented by means of the Stieltjes integral

$$A[f] = \int_a^b f(x) d\alpha(x),$$

where  $\alpha(x)$  is a function of bounded variation (Riesz theorem). It is also known that this functional can be extended to the class  $Q_0$  of piecewise-continuous functions. However, this extension is not unique.

The purpose of the present paper is to give a representation for an arbitrary linear functional defined in  $Q_0$ .

§ 1. Consider a linear functional  $A[f]$ , defined by the conditions

- 1)  $A[f_1 + f_2] = A[f_1] + A[f_2]$ ;
- 2)  $|A[f]| \leq M \sup |f|$

in the class  $Q_0(f)$  of piecewise-continuous functions given on the interval  $(a, b)$ .

Consider the piecewise-continuous functions

$$\eta_\xi(x) = \begin{cases} 1 & \text{if } a < x < \xi; \\ 0 & \text{if } \xi \leq x < b; \end{cases}$$

$$\pi_\xi(x) = \begin{cases} 1 & \text{if } x = \xi; \\ 0 & \text{if } x \neq \xi; \end{cases}$$

and denote

$$\alpha(\xi) = A[\eta_\xi(x)], \quad \sigma(\xi) = A[\pi_\xi(x)].$$

In particular,  $\alpha(b) = A[1]$ , since  $\eta_b(x) = 1$ .

In what follows it will be shown that the functions  $\alpha(\xi)$  and  $\sigma(\xi)$ , which we shall call *characteristic*, uniquely determine the linear functional on  $Q_0$ .

§ 2. **Lemma 1.** There exists no more than a countable number of points  $\zeta_1, \zeta_2, \dots, \dots, \zeta_j, \dots$ , at which  $\sigma(\xi) \neq 0$ , and moreover

$$\sum_{j=1}^{\infty} |\sigma(\zeta_j)| \leq M.$$

We shall call a linear functional  $A[f] = f(\zeta)\sigma(\zeta)$ , where  $a < \zeta < b$ , the *simplest point functional* of the point  $\zeta$ .

A linear functional representable in the form of a sum of simplest point-  
...functionals:

$$A[f] = \sum_{j=1}^{\infty} \sigma_j f(\zeta_j) \left( \sum_{j=1}^{\infty} |\sigma_j| \leq M \right),$$

will be called a **point functional**.

We shall call the linear functional  $\bar{A}[f]$  **regular** if

$$\bar{\sigma}(\xi) = \bar{A}[\pi_\xi(x)] \equiv 0 \quad \text{for every } a < \xi < b.$$

§ 3. We shall show that every linear functional  $A[f]$  can be represented as the sum of a regular and a point linear functional.

Consider the point functional  $A^*[f]$ , equal to

$$A^*[f] = \sum_{j=1}^{\infty} \sigma(\zeta_j) f(\zeta_j),$$

where

$$\sigma(\zeta_j) = A[\pi_{\zeta_j}(x)], \quad \sum_{j=1}^{\infty} |\sigma(\zeta_j)| \leq M.$$

The linear functional

$$\bar{A}[f] = A[f] - A^*[f],$$

as is not difficult to see, is regular and has norm  $\bar{M} \leq 2M$ . Hence the possibility of the representation follows:

$$A[f] = \bar{A}[f] + A^*[f].$$

The characteristic function for  $A^*[f]$  is equal to

$$\alpha^*(\xi) = A^*[\eta_\xi(x)] = \sum_{\zeta_j < \xi} \sigma(\zeta_j),$$

and the characteristic function for  $\bar{A}[f]$  is given by the formula

$$\bar{\alpha}(\xi) = \bar{A}[\eta_\xi(x)] = \alpha(\xi) - \alpha^*(\xi).$$

§ 4. **Lemma 2.** *The function  $\bar{\alpha}(\xi)$  has bounded variation.*

Every function of bounded variation has, at each point, right and left limiting values:

$$\bar{\alpha}_p(\xi) = \bar{\alpha}(\xi + 0), \quad \bar{\alpha}_l(\xi) = \bar{\alpha}(\xi - 0).$$

**Lemma 3.** *There exists at most a countable number of points  $\xi_1, \xi_2, \dots, \xi_i, \dots$ , at which  $\bar{\alpha}_l(\xi_i) \neq \bar{\alpha}(\xi_i)$  or  $\bar{\alpha}(\xi_i) \neq \bar{\alpha}_p(\xi_i)$ .*

It should be noted that  $\alpha(\xi)$  is also a function of bounded variation.

§ 5. Consider the functional

$$\bar{\bar{A}}[f] = \bar{A}[f] - \sum_{\xi_i < \xi} f_p(\xi_i) [\bar{\alpha}_p(\xi_i) - \bar{\alpha}(\xi_i)] - \sum_{\xi_i \leq \xi} f_l(\xi_i) [\bar{\alpha}(\xi_i) - \bar{\alpha}_l(\xi_i)]$$

and its characteristic function

$$\bar{\bar{\alpha}}(\xi) = \bar{\bar{A}}[\eta_\xi(x)] = \bar{\alpha}_p(\xi) - \sum_{\xi_i < \xi} [\bar{\alpha}_p(\xi_i) - \bar{\alpha}_l(\xi_i)]. \quad (1)$$

This formula holds both in the case when  $\xi$  is a point of discontinuity of the function  $\bar{\alpha}(\xi)$ , and in the case when  $\xi$  is a point of continuity and  $\bar{\alpha}(\xi) = \bar{\alpha}(\xi)$ .

**Lemma 4.** *The function  $\bar{\bar{\alpha}}(\xi)$  is continuous for  $a < \xi < b$ , and*

$$\bar{\bar{\sigma}}(\xi) = \bar{\bar{A}}[\pi_\xi(x)] = 0.$$

Let us note that  $\bar{\bar{\alpha}}(\xi)$  has bounded variation  $\bar{\bar{M}} \leq 2\bar{M}$ .

**Lemma 5.** *If, for some regular functional  $\bar{A}[f]$ , the characteristic function  $\bar{\alpha}(\xi)$  is continuous, then in the class  $Q_0(f)$  the representation*

$$\bar{A}[f] = \int_a^b f(x) d\bar{\alpha}(x)$$

holds.

Indeed, divide the interval  $(a, b)$  into parts by the points  $x = x_i$ , including all points of discontinuity of the function  $f(x)$ , and take the step function

$$\bar{f}(x) = f(x_{i-1}), \quad x_{i-1} < x \leq x_i.$$

The difference  $f(x) - \bar{f}(x)$  can be represented in the form

$$f(x) - \bar{f}(x) = \varepsilon(x) + \sum_i [f(x_{i-1}) - f(x_{i-1})] \pi_{x_{i-1}}^\xi(x),$$

where  $\varepsilon(x)$  is a piecewise-continuous function, and  $|\varepsilon(x)| < \varepsilon_0$  for a sufficiently fine mesh. Hence it follows that

$$\left| \bar{A}[f(x)] - \bar{A}[\bar{f}(x)] \right| \leq \bar{M} \varepsilon_0 \quad (\bar{\sigma}(x_i) = 0)$$

or

$$\left| \bar{A}[f] - \sum_i f(x_{i-1}) [\bar{\alpha}(x_i) - \bar{\alpha}(x_{i-1})] \right| \leq M \varepsilon_0.$$

Passing to the limit as  $\Delta x = x_i - x_{i-1} \rightarrow 0$ , we obtain

$$\bar{A}[f] = \lim_{\Delta x \rightarrow 0} \sum_i f(x_{i-1}) [\bar{\alpha}(x_i) - \bar{\alpha}(x_{i-1})] = \int_a^b f(x) d\bar{\alpha}(x).$$

From the construction of  $\bar{\alpha}(x)$  it is clear that this function is the continuous part of the function  $\bar{\alpha}(x)$ .

§ 6. Thus, the following holds.

**Theorem 1.** *Every linear functional  $A[f]$ , defined in the class  $Q_0(f)$  of piecewise-continuous functions  $f(x)$  given on the interval  $(a, b)$ , can be represented in the form*

$$A[f] = \int_a^b f(x) d\bar{\alpha}(x) + \sum_{i=1}^{\infty} \{f(\xi_i) [\bar{\alpha}(\xi_i) - \bar{\alpha}(\xi_i)] + f(\xi_i) [\bar{\alpha}(\xi_i) - \bar{\alpha}(\xi_i)]\} + \sum_{j=1}^{\infty} \sigma(\zeta_j) f(\zeta_j), \quad (2)$$

where

$$\bar{\alpha}(\xi) = \alpha(\xi) - \sum_{\zeta_j < \xi} \sigma(\zeta_j);$$

$\alpha(\xi)$  and  $\sigma(\xi)$  are the characteristic functions of the functional  $A[f]$ ;  $\bar{\alpha}(\xi)$  is the continuous part of the function  $\bar{\alpha}(\xi)$ , computed by formula (1).

Let us note that for a continuous function

$$\bar{A}[f] = \int_a^b f(x) d\bar{\alpha}(x) + \sum_{i=1}^{\infty} f(\xi_i) [\bar{\alpha}_p(\xi_i) - \bar{\alpha}_1(\xi_i)],$$

i.e., the regular functional  $\bar{A}[f]$  is completely determined by the characteristic function  $\alpha(\xi)$  at its points of discontinuity.

§ 7. The linear functional

$$\Gamma[f] = \sum_{j=1}^{\infty} [\omega_j^{(0)} f(\xi_j) + \omega_j^{(1)} f_1(\xi_j) + \omega_j^{(2)} f_p(\xi_j)], \quad \text{where } \omega_j^{(0)} + \omega_j^{(1)} + \omega_j^{(2)} = 0,$$

will be called a **null-functional**.

In the class  $C_0(f)$  a null-functional is always equal to zero. If  $\Gamma = \Gamma_R$  is a regular functional, then  $\omega_j^{(0)} = 0$ ,  $\omega_j^{(1)} = -\omega_j^{(2)}$ , and

$$\Gamma_R[f] = \sum_{j=1}^{\infty} \omega_j^{(2)} [f_p(\xi_j) - f_1(\xi_j)].$$

**Theorem 2.** *The difference of two linear functionals that coincide on  $C_0(f)$  is a null-functional on  $Q_0(f)$ .*

§ 8. We shall call a linear functional  $A[f]$  **nonnegative (positive)** if  $A[f] \geq 0$  for  $f \geq 0$  ( $A[f] > 0$  for  $f \geq \varepsilon > 0$ ).

**Theorem 3.** *For nonnegativity (positivity) of a linear functional  $A[f]$ , it is necessary and sufficient that the following conditions be satisfied:*

- 1) *the characteristic function  $\bar{\alpha}(\xi)$  of the regular part  $\bar{A}[f]$  of the functional  $A[f]$  is a nondecreasing function;*
- 2) *the characteristic coefficients  $\sigma(\zeta_j)$  are nonnegative:  $\sigma(\zeta_j) \geq 0$  (1), (2) and*
- 3)  *$\alpha(b) = A[1] > 0$ .*

§ 9. We shall call linear regular functionals  $A[f]$  and  $B[f]$ , defined on  $Q_0(f)$ , **mutually symmetric** if the condition

$$B[f(x)] = A[f(-x)]$$

is satisfied for any function  $f(x) \in Q_0$  given on the interval  $(a, b)$ .

**Theorem 4.** *The conditions*

$$\alpha(b) = \beta(b), \quad b = -a, \quad \beta(\xi) + \alpha(-\xi) = \alpha(b) = \beta(b)$$

*are necessary and sufficient for the mutual symmetry of  $A[f]$  and  $B[f]$ .*

§ 10. For some applications a representation of linear functionals on  $Q_m(f)$  ( $m \geq 0$ ) is required, where  $Q_m$  is the class of functions piecewise continuous in  $(a, b)$  together with their derivatives up to order  $m$  inclusive. Since the characteristic functions  $\alpha(\xi) = A[\eta_\xi(x)]$  and  $\sigma(\xi) = -A[\pi_\xi(x)]$  of the functional  $A[f]$  are determined by means of functions belonging to the class  $Q_m$ , representation (2) holds for  $A[f]$  defined on  $Q_m(f)$ .

It is not difficult to verify that a linear functional  $A$ , given on  $Q_m$ , can be uniquely extended also to a broader class of functions, for example to the class of functions  $R_{\bar{\alpha}(x)}(f)$  satisfying the following conditions: 1)  $f(x)$  is a bounded function measurable on  $(a, b)$ ; 2)  $f(x)$  has right and left limiting values  $f_1$  and  $f_p$  at all points of discontinuity of the function  $\bar{\alpha}(x)$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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