Geophysics

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1958

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Abstract

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FUNDAMENTAL TYPES OF MOTIONS OF A BAROCLINIC ATMOSPHERE IN THE FIELD OF THE CORIOLIS FORCE

In hydrodynamic weather forecasting, simplified equations of atmospheric dynamics are used, obtained as a result of the "quasistatic" and "quasigeostrophic" approximations (^{1}). These equations are of first order in time, whereas for the original system of equations of atmospheric dynamics the order in time is 5. It follows from this that, in the process of simplifying the original system of equations, certain solutions–namely, solutions corresponding to rapid wave processes –are excluded, or "filtered out" *. In order to justify the validity of such simplifications, it is necessary to determine the character of the rapid wave motions and to indicate the mechanism of "adaptation" of the fields–that is, the way in which an arbitrary state of the atmosphere, defined by 5 independent functions (the velocity vector, pressure, and temperature), passes in a short time into a quasi-equilibrium state, for which the individual fields are coordinated with one another in a definite way**.

In the present note, on the basis of the solution of the problem of small oscillations of a baroclinic atmosphere under sufficiently general assumptions, a classification is given of the fundamental types of dynamical processes in the atmosphere (horizontal vortical motions, gravitational and acoustic waves), and the filtering role of the quasistatic approximation is clarified.

Let us write the system of equations of atmospheric dynamics in the following form:

$$\rho\frac{du}{dt} = -\frac{\partial p}{\partial x} + l\rho v; \qquad \rho\frac{dv}{dt} = -\frac{\partial p}{\partial y} - l\rho u; \qquad \mu\rho\frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g;$$

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right); \qquad \frac{dp}{dt} = c^2\frac{d\rho}{dt}, \tag{1}$$

where x, y, z, t are Cartesian coordinates and time; u, v, w are the velocity components; p is pressure; ρ is density; g is the acceleration of gravity; $l = 2\omega_z$ is the Coriolis parameter (a constant); $c^2 = \varkappa p/\rho$ is the square of the adiabatic speed of sound ($\varkappa = c_p/c_v$ is the ratio of specific heats). In the third equation of dynamics the parameter μ, actually equal to unity, has been introduced. By formally setting $\mu = 0$, we obtain the system of equations in the quasistatic approximation.

We shall take as the "basic state" of the atmosphere a state of relative rest (in the rotating coordinate system), for which $\bar{u} = \bar{v} = \bar{w} = 0$, $\bar{p} = \bar{p}(z)$ and $\bar{\rho} = \bar{\rho}(z)$ are prescribed functions of height, related by the equation statics, and we shall consider small deviations from this state. It is convenient to characterize the perturbed state of the atmosphere by the quantities ψ, φ, Π, χ, and σ, setting

$$\bar{\rho}u = -\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}; \qquad \bar{\rho}v = \frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}; \qquad \bar{\rho}w = \chi; \tag{2}$$

$$\Pi = p - \bar{p}; \qquad \sigma = \rho - \bar{\rho}.$$

Applying the method of linearizing the equations of atmospheric dynamics, i.e., discarding the quadratic terms in them, from (1) one can obtain the following system of equations:

$$\frac{\partial\psi}{\partial t} = -l\varphi; \qquad \frac{\partial\varphi}{\partial t} = l\psi - \Pi; \qquad \frac{\partial\Pi}{\partial t} = -c^2\Delta_1\varphi - \beta\chi - c^2\frac{\partial\chi}{\partial z},$$

$$\mu\frac{\partial\chi}{\partial t} = -\left(\frac{\partial\Pi}{\partial z} + g\sigma\right); \qquad \frac{\partial\sigma}{\partial t} = -\Delta_1\varphi - \frac{\partial\chi}{\partial z}, \tag{3}$$

where $\Delta_1 = \partial^2/\partial x^2 + \partial^2/\partial y^2$; $\beta = (\varkappa - 1)g + dc^2/dz$. Let us note that $\beta = \varkappa R(\gamma_a - \gamma)$ is the principal characteristic of the stratification of the atmosphere $\left(\gamma_a = \frac{\varkappa-1}{\varkappa}\frac{g}{R},\ \gamma = -\frac{\partial T}{\partial z}\right)$. System (3) is of fifth order in time, just as the original system (1), and approximately describes the evolution of perturbations in time*. Boundary conditions with respect to the coordinate z may be specified in the following form:

$$\chi = 0 \qquad \text{for } z = 0; \tag{4}$$

$$\chi \to 0 \qquad \text{for } z \to \infty \tag{5}$$

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* This circumstance was first pointed out by Charney (^{2}).

** The term "adaptation" was introduced in (^{3}) in the consideration of dynamical processes in a two-dimensional model of the atmosphere. An analogous investigation for a three-dimensional atmosphere was carried out in (^{1},^{4}), where the hypothesis of "quasistaticity" was applied *ad hoc.*

(the conditions that the vertical component of the velocity vanish at the lower boundary and that the mass flux vanish at infinity).

In solving the Cauchy problem it is necessary to prescribe 5 initial conditions for $t = 0$:

$$\psi = \psi_0(x, y, z); \qquad \varphi = \varphi_0(x, y, z); \qquad \chi = \chi_0(x, y, z);$$

$$\Pi = \Pi_0(x, y, z); \qquad \sigma = \sigma_0(x, y, z). \tag{6}$$

System (3) admits a family of stationary solutions depending on one arbitrary function of the coordinates $\psi_s(x, y, z)$. For each stationary solution the following relations hold:

$$\Pi_s = l\psi_s; \qquad \sigma_s = -\frac{l}{g}\frac{\partial \psi_s}{\partial z}; \qquad \varphi_s = 0; \qquad \chi = 0. \tag{7}$$

Thus, the stationary solutions are horizontal and nondivergent**. For them the formulas of the geostrophic wind and the equation of statics are valid:

$$u_s = -\frac{1}{l\rho}\frac{\partial \Pi_s}{\partial y}; \qquad v_s = \frac{1}{l\rho}\frac{\partial \Pi_s}{\partial x}; \qquad \frac{\partial \Pi_s}{\partial z} = -g\sigma_s. \tag{8}$$

For system (3) there exists an invariant—a function expressed linearly in terms of the original field characteristics:

$$\widetilde{\Omega} = \Delta_1 \psi - l\left[\sigma - \frac{\partial}{\partial z}\left(\frac{\Pi - c^2\sigma}{\beta}\right)\right]. \tag{9}$$

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* Let us note that quantities which are stationary in the linear theory in reality are quasi-stationary characteristics, i.e., such that their changes in time are determined by the quadratic terms of the original equations. In weather-forecasting theory they play the principal role.

** Let us note that the solution of equations (3) with initial conditions (6) will be stationary only in the case when

$$\varphi_0 = 0, \qquad \Pi_0 = l\psi_0, \qquad \chi = 0, \qquad \sigma_0 = -\frac{l}{g}\frac{\partial \psi_0}{\partial z}.$$

It is easy to verify that, by virtue of equations (3), $\partial\widetilde{\Omega}/\partial t = 0$. The variant we have found for the three-dimensional case is a generalization of the concept of potential vorticity introduced in (3). It is also easy to verify, with the aid of (3) and (4), the invariance of the quantity S^*, defined in the plane $z = 0$:

$$S^* = \Pi^* - c^2\sigma^*, \tag{10}$$

where the asterisk denotes values at $z = 0$. We note that the quantities $\tilde{\Omega}$ and S^* do not contain the parameter μ and, consequently, preserve their invariance also in the transition to the quasistatic approximation ($\mu \to 0$)*.

We shall call **disturbances of wave type** those states for which $\tilde{\Omega} = 0$ under the additional boundary condition $S^* = 0$.

As the principal characteristics of disturbances of wave type it is convenient to consider the quantities φ and χ, which satisfy the system of fourth order:

$$\frac{\partial^2\varphi}{\partial t^2} = c^2\Delta_1\varphi - l^2\varphi + \beta\chi + c^2\frac{\partial\chi}{\partial z};$$

$$\mu\frac{\partial^2\chi}{\partial t^2} = \frac{\partial}{\partial z}\left(\beta\chi + c^2\frac{\partial\chi}{\partial z} + c^2\Delta_1\varphi\right) + g\left(\frac{\partial\chi}{\partial z} + \Delta_1\varphi\right). \tag{11}$$

It is easy to see that when $\mu = 0$ (the hypothesis of quasistaticity) the order of the system is at once lowered by 2 and the number of independent characteristics of the field is correspondingly reduced. System (11) is solved under the boundary conditions (4), (5) and under the initial condition $\varphi = \varphi_0$; $\partial\varphi/\partial t = l\psi_0 - \Pi_0$; $\chi = \chi_0$; $\mu\,\partial\chi/\partial t = -(\partial\Pi_0/\partial z + g\sigma_0)$ (at $t = 0$). Wave solutions possess the property that they disperse, "leaving no trace," i.e., if at the initial moment the characteristics of the wave field differ from zero in some finite region, then as $t \to \infty$ they will tend to zero.

Let us now suppose that the initial characteristics of the field ψ_0, φ_0, Π_0, χ_0, σ_0 satisfy conditions (7) everywhere, except for some finite region where these conditions are violated and the separate characteristics of the field are specified independently of one another. The initial field can be represented as the sum of a stationary and a wave component. With the passage of time the wave disturbance disperses, and in any finite region the field characteristics approach the stationary type corresponding to the stream function ψ_s and pressure $\Pi_s = l\psi_s$. This is the process of adaptation of fields in the atmosphere.

To determine the function ψ_s corresponding to the final state, one may use the equality of $\tilde{\Omega}$ for the initial and final states, which, taking (7) into account, leads to an elliptic equation

$$\Delta_1\psi_s + l^2\frac{\partial}{\partial z}\left[\left(\frac{1}{\beta} + \frac{1}{g}\right)\psi_s + \frac{c^2}{\beta g}\frac{\partial\psi_s}{\partial z}\right] =$$

$$= \Delta\psi_0 - l\left[\sigma_0 - \frac{\partial}{\partial z}\frac{(\Pi_0 - c^2\sigma_0)}{\beta}\right], \tag{12}$$

solved under the boundary condition (invariance of S^*)

$$\psi_s + \frac{c^2}{g}\frac{\partial \psi_s}{\partial z} = \frac{1}{l}\left(\Pi_0 - c^2\sigma_0\right) \quad \text{for } z = 0. \tag{13}$$

The differential operator on the left-hand side of (12) occurs in the theory of changes of the baric field in a baroclinic atmosphere $(1, 4)$.

An estimate of the characteristic time for the process of adaptation requires a more detailed analysis of wave disturbances in the atmosphere. System (11) is solved by the method of separation of variables; moreover, for the isothermal atmosphere ($c^2 = \text{const}$, $\beta^2 = \text{const}$) the solution is obtained as a superposition of particular solutions of the form $\exp\left[-\frac{\beta+g}{2c^2}z + i(k_1x + k_2y + mz - \omega t)\right]$. The frequency ω is determined for each triple of wave numbers (k_1, k_2, m) from a fourth-order characteristic equation; moreover, for ω^2 two values are obtained, corresponding to two types of waves:

$$\omega_a^2,\ \omega_g^2 = \frac{c^2}{2}\left[k^2 + \frac{l^2}{c^2} + \frac{m^2}{\mu} + \frac{1}{\mu}\left(\frac{\beta+g}{2c^2}\right)^2\right] \times$$

$$\times \left\{1 \pm \left[1 - \frac{4g\beta}{\mu c^4}\,\frac{k^2 + \left[m^2 + \left(\frac{\beta+g}{2c^2}\right)^2\right]\frac{c^2 l^2}{g^3}}{\left[k^2 + \frac{l^2}{c^2} + \frac{m^2}{\mu} + \frac{1}{\mu}\left(\frac{\beta+g}{2c^2}\right)^2\right]^2}\right]\right\}.$$

The plus sign corresponds to the branch ω_a^2, the minus sign to ω_g^2. It is natural to call waves with high frequencies (propagation speed) ω_a acoustic waves, and the slower oscillations with frequencies ω_g gravity waves.*

In addition to solutions with $m \neq 0$, which it is natural to call **internal waves**, there exist special solutions for $m = 0$, for which $\chi \equiv 0$, $\varphi \neq 0$, with frequency spectrum $\omega^2 = k^2c^2 + l^2$. These waves may be called **two-dimensional**; they adjoin the acoustic branch of atmospheric oscillations.

In the limiting transition $\mu \to 0$ (the quasistatic approximation), internal acoustic waves disappear (the frequencies go to infinity), while the frequencies of gravity waves increase somewhat; however, this change of the spectrum is insignificant for large-scale disturbances (the horizontal scale is much greater than the height of the homogeneous atmosphere). Thus the quasistatic approximation is a means of filtering internal acoustic waves.

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* The corresponding invariants in the quasistatic approximation were found by A. S. Monin (4).

The process of adaptation of the atmosphere to the quasistatic state occurs over a period of the order of the time required for a sound wave to traverse the main depth of the atmosphere**, which is several minutes. After static equilibrium has been reached, the process of adaptation of the atmosphere to the state of geostrophic equilibrium continues; moreover, on average over the depth of the atmosphere such a state is established after the scattering of two-dimensional waves, and somewhat later, after the scattering of slower internal gravity waves, geostrophic equilibrium is established at all heights. The second stage of the adaptation process can be investigated with the aid of approximate "quasistatic" equations, as was done in the work of A. S. Monin (4).

Received
30 V 1958

CITED LITERATURE

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* Oscillations of the acoustic type depend mainly on the compressibility of the medium (c^2), and to a lesser degree on Archimedean forces (β). For oscillations of the gravity type the principal factor is stratification, although the compressibility of the medium also has an influence. For the case $l = 0$ and $\varkappa \to \infty$, which corresponds to an incompressible stratified medium, for the frequency ω_g^2 one obtains in the limit the well-known formula (5)

$$\omega_g^2 = \frac{g}{H}\frac{k^2}{k^2+m^2+1/4H^2} \qquad \left(H = \frac{p_0}{g\rho_0}\right).$$

** In 1 min. a sound wave traverses a layer 20 km thick.

Note: Figure translations are in progress. See original paper for figures.

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