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Abstract

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On the Separation of Singularities of Analytic Functions

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In the present note we prove several theorems on the decomposition of an analytic function into a sum of functions having, as their set of singular points, a prescribed part of the set of singularities of the original function. In doing so it proves convenient to apply certain considerations from the theory of linear topological spaces.

Theorem 1. *Let F_0, F_1, F_2 be closed sets of the extended complex plane; let G_0, G_1, G_2 be their open complements, with $F_0 = F_1 \cup F_2$. If the function $u_0(z)$ is analytic in G_0, then there exist functions $u_1(z)$ and $u_2(z)$ such that in G_0*

$$u_0(z) = u_1(z) + u_2(z);$$

$u_i(z)$ is analytic in G_i $(i = 1, 2)$.

Remark. Everywhere below we shall assume that the functions under consideration are equal to zero at the point at infinity, whenever it is contained in their domain of definition.

In the case when F_0 is the real line, $F_1 = [-1, +1]$, $F_2 = (-\infty, -1] \cup [+1, +\infty)$, this theorem was proved by Poincaré ([1,2]).

Lemma. *Let X_0, X_1, X_2 be locally convex Hausdorff linear topological spaces; $X_0 \subset X_1$, $X_0 \subset X_2$ (in the sense that X_0 is algebraically isomorphic to a part of X_i, $i = 1, 2$); the topology in X_0 is the least upper bound of the topologies induced in X_0 from X_1 and X_2. If f_0 is an additive continuous functional given in X_0, then there exist two functionals f_1 and f_2 such that $f_0(x) = f_1(x) + f_2(x)$ $(x \in X_0)$, and the functional f_i is continuous in the topology induced on X_0 from X_i $(i = 1, 2)$.*

Proof. Consider the product of the spaces $X_1 \times X_2$ and in it the closed linear subspace $\widetilde{X}_0$, consisting of all pairs of the form (x, x), where $x \in X_0$. Let $\tilde{f}((x, x)) = f(x)$. It is easy to see that the functional $\tilde{f}$ is continuous in the topology induced on $\widetilde{X}_0$ from $X_1 \times X_2$. Let $\tilde{f}$ be a linear functional extending

$\tilde{f}$ from $\widetilde{X}_0$ to all of $X_1 \times X_2$ with preservation of continuity (^{3}). Put now $f_1(x) = \tilde{f}((x, 0))$, $f_2(x) = \tilde{f}((0, x))$. The lemma is proved.

Now let X_{F_0} (X_{F_1}, X_{F_2}) be the totality of all functions analytic on F_0 (respectively F_1, F_2). Let $g_n^{(j)}$ ($j = 0, 1, 2$) be a decreasing sequence of open sets converging to F_j as to a kernel; let $B_n^{(j)}$ be the normed space of functions analytic and bounded in $g_n^{(j)}$, with norm

$$\|f\|_n^{(j)} = \sup_{z \in g_n^{(j)}} |f(z)|.$$

Then $B_n^{(j)} \subset B_{n+1}^{(j)}, n = 1, 2, \dots,$

$$X_{F_j} = \bigcup_{n=1}^{\infty} B_n^{(j)}.$$

Let in X_{F_j} there be introduced the topology of the inductive limit of the sequence of spaces $B_n^{(j)}$ (^{4}). Then the spaces $X_j = X_{F_j}$ turn out to be the so-called spaces of type $(LN)^*$ (^{5,6}).

We turn to the proof of Theorem 1. Let

$$f_0(x) = \frac{1}{2\pi i} \int_{C_x} x(z) u_0(z)\, dz,$$

where $x \in X_{F_0}$; C_x is a finite system of closed simple rectifiable pairwise nonintersecting contours forming the boundary of a domain containing F_0 inside, and $x(z)$ is analytic in the closure of this domain. The functional $f_0(x)$ is continuous in X_{F_0}. Put

$$\psi_{\tilde{z}}(z) = \frac{1}{z - \tilde{z}}.$$

If $\tilde{z} \in F_0$, then $\psi_{\tilde{z}} \in X_{F_0}$, and

$$u_0(\tilde{z}) = f_0(\psi_{\tilde{z}}).$$

Using the lemma, we find functionals f_1 and f_2, continuous respectively in the spaces X_1 and X_2, such that

$$u(\tilde{z}) = f_0(\psi_{\tilde{z}}) = f_1(\psi_{\tilde{z}}) + f_2(\psi_{\tilde{z}}).$$

Let $u_j(\tilde{z}) = f_j(\psi_{\tilde{z}})$, $\tilde{z} \in G_j$ $(j = 1, 2)$. Using the continuity of the functional f_j in the space X_{F_j}, it is not difficult to prove the existence of the derivative $u_j'(\tilde{z})$ at all points of G_j. The theorem is proved.

We note a consequence of Theorem 1.

Theorem 2. *Let G_1 and G_2 be two domains having a nonempty intersection $G_1 \cap G_2$. Suppose that the system of regular functions $\{\theta_k^{(j)}(z)\}_{k=1}^{\infty}$ $(j = 1, 2)$ is complete in the domain G_j (in the sense of uniform convergence on compact subsets of G_j). Then the system $\{\theta_k^{(1)}(z), \theta_k^{(2)}(z)\}_{k=1}^{\infty}$ is complete in $G_1 \cap G_2$.*

Indeed, if the function $u(z)$ is analytic in $G_1 \cap G_2$, then, by Theorem 1, $u(z) = u_1(z) + u_2(z)$, where $u_k(z)$ is analytic in G_k. Hence it is clear that the linear span of the system $\{\theta_k^{(1)}(z), \theta_k^{(2)}(z)\}_{k=1}^{\infty}$ is dense in the set of functions analytic in $G_1 \cap G_2$.

In Theorem 1 nothing is said about the behavior of the functions u_1 and u_2 near their sets of singularities. An idea of their properties can sometimes be obtained by knowing the behavior of the function $u_0(z)$ itself.

Theorem 3. *Let $u_0(z)$ be a function analytic in the disk $|z| < 1$ and such that:*

$$1) \int_0^{\theta} |u_0(re^{i\varphi})|^p \, d\varphi < C < \infty, \quad p > 1; \quad \theta \text{ is some number from the interval } (0, 2\pi];$$

$$2) \text{ the integrals } \int_0^1 |u_0(r)| \, dr, \quad \int_0^1 |u_0(re^{i\theta})| \, dr \text{ converge.}$$

Then there exists a function $\Psi(\varphi)$, $0 \le \varphi \le \theta$, such that

$$\int_0^{\theta} |\Psi(\varphi)|^p \, d\varphi < +\infty,$$

$$u_0(z) = \int_0^{\theta} \frac{\Psi(\varphi) \, d\varphi}{e^{i\varphi} - z} + u_1(z), \quad |z| < 1,$$

where $u_1(z)$ is a function regular everywhere outside the arc $z = e^{i\varphi}$, $\theta \le \varphi \le 2\pi$.

Proof. Let X_0 be the set of all functions analytic in the closed exterior of the unit circle and equal to zero at the infinitely distant point. Put further: X_1 is the set of functions analytic on the arc $z = e^{i\varphi}$, $\theta \le \varphi \le 2\pi$; X_2 is the set of all complex-valued functions summable on the arc $z = e^{i\varphi}$, $0 \le \varphi \le \theta$, with exponent $q = \frac{p}{p-1}$. Let B_n be the Banach space consisting of functions analytic in the closure of the domain whose boundary is ⋯

the curve Γ_n, consisting of the arcs $z = \left(1 - \frac{1}{n}\right) e^{i\varphi}$, $z = \left(1 + \frac{1}{n}\right) e^{i\varphi}$, $\theta - \varepsilon_n < \varphi < 2\pi + \varepsilon_n$, and the segments $z = te^{i(\theta-\varepsilon_n)}$, $z = te^{i(2\pi+\varepsilon_n)}$, $1 - \frac{1}{n} \leq t \leq 1 + \frac{1}{n}$; here $\{\varepsilon_n\}$ is a sequence of positive numbers monotonically decreasing to zero; the norm in B_n is

$$\|x\|_n = \max_{z\in\Gamma_n} |x(z)|.$$

In X_1 we introduce the topology of the inductive limit of the sequence of spaces B_n. In X_2 we introduce a norm by setting

$$\|x\|_{X_2} = \left(\int_0^\theta |x(e^{i\varphi})|^q \, d\varphi\right)^{1/q}, \qquad X_0 \subset X_1, \quad X_0 \subset X_2,$$

in the sense that every function regular in the set $|z| \geq 1$, regular on the arc $z = e^{i\varphi}$, $\theta \leq \varphi \leq 2\pi$, generates on the arc $z = e^{i\varphi}$, $0 \leq \varphi \leq \theta$, a function summable to the power q.

Introduce in X_0 the topology which is the exact upper bound of the topologies induced from X_1 and X_2.

Define a functional on X_0 by putting

$$f_0(x) = \frac{1}{2\pi i} \int_0^{2\pi} x(re^{i\varphi})\, u_0(re^{i\varphi})\, dre^{i\varphi} \qquad (0 < r < 1),$$

where $x \in X_0$, and moreover $x(z)$ is a function analytic up to the circle $|z| = r$. The functional f_0 is continuous in the topology X_0. Let us prove this.

$$|f_0(x)| \leq \left|\frac{1}{2\pi i} \int_0^\theta x(re^{i\varphi})\, u_0(re^{i\varphi})\, dre^{i\varphi}\right| + \left|\frac{1}{2\pi i} \int_\theta^{2\pi} x(re^{i\varphi})\, u_0(re^{i\varphi})\, dre^{i\varphi}\right| \leq$$

$$\leq \frac{1}{2\pi} C^{1/p} \left(\int_0^\theta |x(re^{i\varphi})|^q \, d\varphi\right)^{1/q} + \left|\frac{1}{2\pi i} \int_\theta^{2\pi} x(re^{i\varphi})\, u_0(re^{i\varphi})\, dre^{i\varphi}\right|. \qquad (*)$$

If $r \to 1 - 0$, then the first term on the right-hand side tends to

$$\frac{1}{2\pi} C^{1/p} \|x\|_{X_2}.$$

Consider the second term. Let $1 > r_1 > r$. Then

$$\int_\theta^{2\pi} x(re^{i\varphi})\, u_0(re^{i\varphi})\, dre^{i\varphi} - \int_\theta^{2\pi} x(r_1e^{i\varphi})\, u_0(r_1e^{i\varphi})\, dr_1e^{i\varphi} -$$

$$- \int_r^{r_1} x(\zeta)\, u_0(\zeta)\, d\zeta + \int_r^{r_1} x(\zeta e^{i\theta})\, u_0(\zeta e^{i\theta})\, d\zeta e^{i\theta} = 0.$$

Therefore

$$\left|\int_{\theta}^{2\pi} x(re^{i\varphi})\,u_0(re^{i\varphi})\,dre^{i\varphi} - \int_{\theta}^{2\pi} x_i'(r_1e^{i\varphi})\,u_0(r_1e^{i\varphi})\,dr_1e^{i\varphi}\right| \le$$

$$\le \max_{r<\zeta<r_1}\{\,|x(\zeta)|,\ |x(\zeta e^{i\varphi})|\,\}\left[\int_r^{r_1}|u_0(\zeta e^{i\theta})|\,d\zeta + \int_r^{r_1}|u_0(\zeta)|\,d\zeta\right].$$

From condition 2) of Theorem 3 we conclude that there exists the limit

$$\lim_{r\to 1-0}\int_{\theta}^{2\pi} x(re^{i\varphi})\,u_0(re^{i\varphi})\,dre^{i\varphi} = f^*(x).$$

Moreover, if $x \in B_n$, then

$$\left|\int_0^{2\pi} x\left(re^{i\varphi}\right)u_0\left(re^{i\varphi}\right)\,dre^{i\varphi} - f^*(x)\right| \le \|x\|_n K,$$

i.e.

$$|f^*(x)| \le K\|x\|_n + \left|\int_0^{2\pi} x\left(re^{i\varphi}\right)u_0\left(re^{i\varphi}\right)\,dre^{i\varphi}\right| \le$$

$$\le K\|x\|_n + \left\{\int_0^{2\pi}\left|u_0\left(\left(1-\frac{1}{n}\right)e^{i\varphi}\right)\right|\,d\varphi\right\}\|x\|_n.$$

Here

$$K = \int_0^1 |u_0(r)|\,dr + \int_0^1\left|u_0\left(re^{i\theta}\right)\right|\,dr.$$

Letting now r tend to 1 in inequality $(*)$, we obtain

$$|f_0(x)| \le \frac{C^{1/p}}{2\pi}\|x\|_{X_2} + \frac{1}{2\pi}K_1^{(n)}\|x\|_n \qquad (x \in B_n).$$

This inequality shows that the functional f_0 is continuous in the topology induced on $B_n \cap X_0$ by the topologies of B_n and X_2. From a theorem of B. M. Makarov[7] it follows that f_0 is continuous in the topology of X_0. Applying the lemma, we obtain that

$$f_0(x) = f_1(x) + f_2(x) \qquad (x \in X_0),$$

where f_1 is continuous in X_1, and f_2—in X_2. But

$$u_0(\tilde{z}) = f_0(\psi_{\tilde{z}}) = f_1(\psi_{\tilde{z}}) - f_2(\psi_{\tilde{z}}) \qquad (|\tilde{z}| < 1).$$

It is well known that

$$f_2(x) = \int_0^\theta \Psi(\varphi) x(e^{i\varphi})\, d\varphi,$$

where the function Ψ is such that

$$\int_0^\theta |\Psi(\varphi)|^p\, d\varphi < +\infty.$$

At the same time, $\psi_{\tilde{z}}$ is an element of X_1, if $\tilde{z} \neq e^{i\varphi}$, $0 \leq \varphi \leq 2\pi$. Hence it is easy to conclude, as in Theorem 1, that

$$u_1(\tilde{z}) = f_1(\psi_{\tilde{z}})$$

is a function regular outside the arc $\tilde{z} = e^{i\varphi}$, $0 \leq \varphi \leq 2\pi$. The theorem is proved.

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