MATHEMATICS

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1958

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Abstract

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MATHEMATICS

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ON THE CONVERGENCE OF FOURIER SERIES WITH RESPECT TO SYSTEMS OF THE TYPE $\{\varphi(nx)\}$, CLOSE TO THE TRIGONOMETRIC SYSTEM

(Presented by Academician A. N. Kolmogorov, 10 VII 1957)

Let $L_2'[-\pi,\pi]$ be the space of odd complex-valued functions whose modulus squared is summable on $[-\pi,\pi]$. We shall regard every function as periodically extended with period 2π to the entire real line. Given a function $\varphi(x)$, construct the system of functions $\{\varphi(nx)\}$, $n = 1, 2, \ldots$. Questions of completeness of systems of this kind have been considered by many authors ($^{1-5}$).

Consider the Fourier series of the function $\varphi(x)$ with respect to the trigonometric system $\{\sin nx\}$

$$\varphi(x) \sim \sum_{k=1}^{\infty} b_k \sin kx. \tag{1}$$

If $b_1 \neq 0$, then it is not difficult to construct a system $\{g_n(x)\}$ conjugate to the system $\{\varphi(nx)\}$, i.e. a system for which $(\varphi(nx), g_k(x)) = \delta_{nk}$. The system $\{g_n(x)\}$ has the form

$$g_n(x) = \sum_{k=1}^{n} \lambda_{nk} \sin kx.$$

For each function $F(x) \in L_2'[-\pi,\pi]$ one can construct the Fourier series with respect to the system $\{\varphi(nx)\}$

$$F(x) \sim \sum_{k=1}^{\infty} a_k \varphi(kx), \tag{2}$$

where the coefficients a_k are determined by the formulas

$$a_k = \int_{-\pi}^{\pi} F(x) g_k(x)\, dx.$$

The problem of the present note is to find conditions that must be imposed on the function $F(x)$ and on the system $\{\varphi(nx)\}$ in order that the Fourier series of this function with respect to the system $\{\varphi(nx)\}$ converge uniformly, everywhere, almost everywhere, or in the mean to $F(x)$.

Consider the series with respect to the system $\{\sin nx\}$ having the same coefficients as the series (2),

$$\sum_{k=1}^{\infty} a_k \sin kx. \tag{3}$$

Suppose that the coefficients of the series (1) satisfy the condition

$$\sum_{k=1}^{\infty} |b_k| < +\infty;$$

then

$$\varphi(x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

Denote the partial sum of the series (2) by $S_n(x)$, and the partial sum of the series (3) by $\tau_n(x)$. The relation holds

$$S_n(x) = \sum_{k=1}^{\infty} b_k \tau_n(kx). \tag{4}$$

Suppose that the series (2) converges in mean to the function $F(x)$, and the series (3) converges in mean to some function $f(x) \in L_2'[-\pi, \pi]$. Then from equality (4) it follows that $F(x)$ and $f(x)$ are connected by the relation

$$F(x) = \sum_{k=1}^{\infty} b_k f(kx). \tag{5}$$

Thus, the question of the mean convergence of the series (2) to $F(x)$ can be studied by means of the functional equation (5), where $F(x)$ is the given function; $\{b_k\}$ are the given numbers and $f(x)$ is the sought function.

In order to formulate conditions imposed on the coefficients $\{b_k\}$ that are sufficient for solving equation (5), we shall use the well-known notion of a multiplicative function, as well as certain theorems obtained by V. Ya. Kozlov (^{5}).

Let the integer $n > 0$ have the factorization into prime factors

$$n = p_1^{\alpha_1(n)} \cdot p_2^{\alpha_2(n)} \cdots p_{k_n}^{\alpha_{k_n}(n)}, \tag{6}$$

where $p_1, p_2, \dots, p_i, \dots$ are the prime numbers arranged in increasing order. To each prime number p_i we assign the complex variable z_i. To the number n having the factorization (6), we assign the function

$$\mu(n) = z_1^{\alpha_1(n)} \cdot z_2^{\alpha_2(n)} \cdots z_{k_n}^{\alpha_{k_n}(n)} \qquad (n = 2, 3, 4, \dots) \quad (\mu(1) = 1).$$

The function $\mu(n)$ is a multiplicative function of n.

Consider the set of sequences of complex numbers $z_1, z_2, \dots, \dots, z_i, \dots$, satisfying the conditions: 1) $|z_i| < 1$; 2) $\sum_{i=1}^{\infty} |z_i|^2 < +\infty$.

We shall denote this set by G. A sequence of complex numbers $\{z_1, z_2, \dots, z_i, \dots\}$ will be called a point in the domain G.

Definition (^{5}). A **kernel** is a function of x and of a countable set of complex variables $z_1, z_2, \dots, z_i, \dots$, defined in the following way:

$$M(x; z_1, z_2, \dots, z_i, \dots) = \sum_{n=1}^{\infty} \mu(n) \sin nx. \tag{7}$$

Lemma (V. Ya. Kozlov (^{5})). For each fixed point of the domain G, the series (7) converges in mean and the kernel $M(x; z_1, z_2, \dots, z_i, \dots)$ belongs to $L_2'[-\pi, \pi]$.

Definition (^{5}). A function of a countable set of complex variables $\{z_1, z_2, \dots, z_i, \dots\}$ is called the **conjugate function of the element** $\varphi(x)$, if it is defined in the following way:

$$\Phi^{\varphi}\{z_1, z_2, \dots, z_i, \dots\} = \frac{1}{\pi} \int_{-\pi}^{\pi} M(x; z_1, z_2, \dots, z_i, \dots) \overline{\varphi(x)}\, dx.$$

Take for the function $\varphi(x) \in L_2'[-\pi, \pi]$ the Fourier series $\varphi(x) = \sum_{k=1}^{\infty} b_k \sin kx$. Then the conjugate function for the element $\varphi(x)$ will have the form

$$\Phi^{\varphi}(z_1, z_2, \dots, z_i, \dots) = \sum_{k=1}^{\infty} b_k^* \mu(k). \tag{8}$$

Thus, to each function $\varphi(x) \in L_2'[-\pi, \pi]$ one can assign the series

$$\sum_{k=1}^{\infty} b_k \mu(k). \tag{9}$$

Consider the set of sequences of complex numbers $\{z_1, z_2, \dots, z_i, \dots\}$ satisfying the condition $|z_i| \le p_i^{1+\varepsilon}$ $(\varepsilon > 0)$, where p_i is the i-th prime number. We shall denote this set by G_1. Obviously, $G \subset G_1$.

Definition. Let a function $\varphi(x) \in L_2'[-\pi, \pi]$ be given,

$$\varphi(x) = \sum_{k=1}^{\infty} b_k \sin kx.$$

The system $\{\varphi(nx)\}$ is called **close to the trigonometric system** $\{\sin nx\}$, or a T-system, if the coefficients $\{b_n\}$ satisfy the conditions:

$$\text{The series (9) converges absolutely in the domain } G_1;$$

$$\text{The series (9) has no zeros in the domain } G_1. \tag{10}$$

Remark. Conditions (10) are fulfilled, for example, in the case when

$$|b_1| > \sum_{k=2}^{\infty} |b_k|\, k^{1+\varepsilon}. \tag{11}$$

The converse is not true, as the following example shows: there exist functions $\varphi(x)$ for which conditions (10) are fulfilled, while condition (11) is not.

Example 1.

$$\varphi(x) = \sum_{k=0}^{\infty} \frac{\sin 2^k x}{k!} \quad (0! = 1); \qquad \sum_{k=1}^{\infty} b_k \mu(k) = \sum_{k=0}^{\infty} \frac{\mu(2^k)}{k!} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

It is clear that here conditions (10) are fulfilled, but condition (11) is not fulfilled.

We shall now formulate conditions sufficient for the solution of the functional equation (5).

Lemma 1. *In order that the functional equation*

$$F(x) = \sum_{k=1}^{\infty} b_k f(kx),$$

where $F(x)$ is a given function from $L_2'[-\pi, \pi]$, $b_1, b_2, \dots, b_n, \dots$ are given numbers, and $f(x)$ is an unknown function, have a solution, it is sufficient that the series defining the function of a countable set of complex variables

$$\Phi(z_1, z_2, \dots, z_i, \dots) = \sum_{k=1}^{\infty} b_k \mu(k)$$

converge absolutely in the domain $G_1\{|z_i| \leq p_i^{1+\varepsilon}\}$*, and that the function* $\Phi(z_1, z_2, \dots, z_i, \dots)$ *have no zeros in the same domain. Under these conditions the solution can be represented in the form*

$$f(x) = \sum_{k=1}^{\infty} A_k F(kx),$$

where the coefficients A_k *satisfy the condition*

$$\sum_{k=1}^{\infty} |A_k| \, k^{\varepsilon_1} < +\infty \qquad (\varepsilon_1 < \varepsilon).$$

With the aid of Lemma 1 one can obtain the following results.

Theorem 1. Whatever the function $F(x) \in L_2'[-\pi, \pi]$, the Fourier series of this function with respect to the T-system $\{\varphi(nx)\}$ converges in mean to $F(x)$, and the series formed from the sum of the squares of the Fourier coefficients of this function converges.

Corollary. It follows from Theorem 1 that the T-system $\{\varphi(nx)\}$ is a complete system in the space $L_2'[-\pi, \pi]$ and that, moreover, it is a Riesz basis in this space.

Theorem 2. In order that the Fourier series (2) of a continuous function $F(x)$ with respect to the T-system $\{\varphi(nx)\}$ converge uniformly to $F(x)$, it is necessary and sufficient that the corresponding trigonometric series (3) converge uniformly.

It follows in particular from Theorem 2 and Lemma 1 that if the function $F(x)$ satisfies a Lipschitz condition of order α $(0 < \alpha \leqslant 1)$, then the Fourier series of this function with respect to the T-system converges uniformly to this function.

Theorem 3. In order that the Fourier series (2) of a continuous function $F(x)$ with respect to the T-system $\{\varphi(nx)\}$ converge at every point to $F(x)$ and that its partial sums be uniformly bounded, it is necessary and sufficient that the corresponding trigonometric series (3) converge at every point to the continuous function $f(x)$ and that its partial sums be uniformly bounded.

Theorem 4. In order that the Fourier series (2) of a function $F(x) \in L_2'$ with respect to the T-system $\{\varphi(nx)\}$ converge almost everywhere to $F(x)$ and that its partial sums have an integrable majorant, it is necessary and sufficient that the corresponding trigonometric series (3) converge almost everywhere to $f(x) \in L_2'[-\pi, \pi]$ and that its partial sums have an integrable majorant.

It follows from Theorem 4 that, for series of the form $\sum_{k=1}^{\infty} a_k \varphi(kx)$ with respect to the T-system $\{\varphi(nx)\}$, the Kolmogorov-Seliverstov-Plessner theorem is valid.

Theorem 5. Let a T-system $\{\varphi(nx)\}$ be given. If the series

$$\sum_{k=2}^{\infty} a_k^2 \ln k$$

converges, then the series (2) converges almost everywhere. In particular, if the series (2) is the Fourier series of a function $F(x) \in L_2'$, then it converges almost everywhere to $F(x)$.

A similar result, asserting only almost-everywhere convergence of the series (2) for systems $\{\varphi(nx)\}$ with the condition

$$\sum_{k=1}^{\infty} |b_k| < +\infty,$$

which does not ensure completeness of the system $\{\varphi(nx)\}$, was obtained by V. F. Gaposhkin*.

For summation of the series (2) by the Toeplitz method, theorems analogous to Theorems 2, 3, and 4 are valid. In particular, the following theorem holds:

Theorem 6. Whatever the odd, everywhere continuous, periodic function $F(x)$ with period 2π, the Fourier series of this function with respect to the T-system $\{\varphi(nx)\}$, by the $(C, 1)$ method, is uniformly summable to this function.

Received
10 VII 1957

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* Unpublished.

Note: Figure translations are in progress. See original paper for figures.

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