MATHEMATICS

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Abstract

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ON THE ASYMPTOTIC BEHAVIOR OF ANALYTIC FUNCTIONS

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The paper indicates a certain dependence between the asymptotic behavior

$$f(z) = \sum_{s=1}^{\infty} (-1)^{s-1} a_s z^s = a_1 z - a_2 z^2 + a_3 z^3 - \cdots \tag{1}$$

as $z \to \infty$ along the positive real axis and the analytic properties of the function $a_s = a(s)$, which is considered chiefly as a rational function of the complex variable s. For this case $f(z)$ can be analytically continued to the whole half-axis $x \geqslant 0$. If in the half-plane $\mathrm{Re}(s) \geqslant 0$ the function a_s has no poles, then $f(x)$ tends to a finite limit. If a_s has poles in this half-plane, then $f(x)$ grows without bound, and the order of growth depends on the abscissa of the rightmost pole. Knowing the poles of a_s, one can indicate not only the order of growth, but also an exact asymptotic formula for $f(x)$. This dependence, and moreover in the very same form, turns out to be valid not only for rational functions a_s. Here we restrict ourselves to additional indications only for more or less special classes of functions for which the properties mentioned hold.

Definition. The function a_s is said to possess **property A** if it is defined at the integer points $0, 1, 2, \ldots$, the series (1) converges in some neighborhood of the point $z = 0$, $f(z)$ is analytically continuable to the whole half-axis $x \geqslant 0$, and tends to a_0 as $x \to \infty$.

Lemma. If the n-th difference $\Delta^n a_s$ at the point 0 for the function a_s tends to zero as $n \to \infty$, then a_s and $a_s/\Gamma(s+1)$ possess property A.

Theorem 1. If a_s is a rational function having no poles in the half-plane $\mathrm{Re}(s) \geqslant 0$, then a_s and $a_s/\Gamma(s+1)$ both possess property A.

Proof. We shall show that $\Delta^n a_0 \to 0$ as $n \to \infty$. For a polynomial the assertion is obvious. In the presence of poles it is sufficient to consider the case $a_s = (s-p)^{-k}$, where $\mathrm{Re}(p) < 0$. Denote $p = -q$, $\mathrm{Re}(q) > 0$; setting $F_n(x, q) = x^{k-1} e^{-qx} (1 - e^{-x})^n$, we have for $n \geqslant 0$

$$a_n = \frac{1}{(k-1)!}\int_0^\infty x^{k-1}e^{-(q+n)x}\,dx, \qquad \Delta^n a_0 = \frac{(-1)^n}{(k-1)!}\int_0^\infty F_n(x,q)\,dx.$$

Take arbitrary $x_0 > 0$; let

$$I_1 = \int_0^{x_0} F_n(x,q)\,dx, \qquad I_2 = \int_{x_0}^\infty F_n(x,q)\,dx.$$

Then

$$|\Delta^n a_0| \leqslant \frac{1}{(k-1)!}(|I_1| + |I_2|).$$

Let an arbitrary $\varepsilon > 0$ be given. Since, for any n,

$$|I_2| \leqslant \int_{x_0}^\infty F_n(x,\operatorname{Re}(q))\,dx < \int_{x_0}^\infty x^{k-1}e^{-x\operatorname{Re}(q)}\,dx,$$

x_0 can be chosen so large that, for all n, the inequality $|I_2| < \varepsilon$ will hold. In the integral I_1 we have $0 \leqslant x \leqslant x_0$, $0 \leqslant 1 - e^{-x} \leqslant 1 - e^{-x_0} = \lambda < 1$; consequently,

$$|I_1| \leqslant \lambda^n \int_0^{x_0} x^{k-1}e^{-x\operatorname{Re}(q)}\,dx,$$

where the second factor on the right-hand side does not depend on n. Hence, for all sufficiently large n, $|I_1| < \varepsilon$, and then

$$|\Delta^n a_0| < \frac{2\varepsilon}{(k-1)!}.$$

Theorem 2. *Every entire function a_s of order 1 and type $\sigma < \pi$ has property A.*

Proof. From the inequality $|a_s| < Me^{\sigma|s|}$ we conclude for the series (1) that its radius of convergence is $R \geqslant e^{-\sigma}$. Denote by $\psi(s)$ the function associated with a_s by Borel. It is known that $\psi(s)$ is regular outside the circle $|s| = \sigma$, and a_s is represented in the whole plane by means of it by the integral

$$a_s = \frac{1}{2\pi i}\int_C e^{s\zeta}\psi(\zeta)\,d\zeta, \tag{2}$$

where the contour C may be taken to be the circle $|\zeta| = r$, $\sigma < r < \pi$.

From (1) and (2) it follows that there is an $x_0 > 0$ such that, for $0 \leqslant x \leqslant x_0$, one can write

$$f(x) = a_0 - \frac{1}{2\pi i} \int_C \psi(\zeta) \frac{d\zeta}{1 + xe^{\zeta}}. \tag{3}$$

For fixed $x > 0$, the poles of the expression $(1 - xe^{\zeta})^{-1}$ will be $\zeta = -\ln x + (2k+1)\pi i$ $(k = 0, \pm 1, \pm 2, \ldots)$. They all lie on a straight line perpendicular to the real axis, and as x increases from 0 to ∞, the abscissa of these poles will change from ∞ to $-\infty$. In doing so, the poles move parallel to the real axis and on their path do not meet the contour C. It follows that the right-hand side of (3) is a regular function of x not only for $0 \leqslant x \leqslant x_0$, but also on the entire half-axis $x \geqslant 0$. Directly from (3) it follows that $f(x) \to a_0$ as $x \to \infty$.

In this theorem the condition $\sigma < \pi$ cannot be replaced by a weaker one, since for the function $a_s = \frac{1}{s} \sin \pi s$ $(\sigma = \pi)$ the assertion of the theorem is already false. But low growth is not in itself a necessary condition for the validity of property A. Thus, for example, the function $1/\Gamma(s+1)$ obviously has this property, and yet its type $\sigma = \infty$. In view of the special role of the factor $1/\Gamma(s+1)$ in the composition of the coefficients of a power series, we give one theorem whose proof is analogous to the preceding one.

Theorem 3. *Let a_s be an entire function of order 1 and type $\sigma < \pi/2$. Then not only a_s, but also $a_s/\Gamma(s+1)$, has property A.*

The class of functions a_s for which the validity of property A can be asserted admits further extensions. We state without proof the following "multiplication" theorem.

Theorem 4. *If $\varphi(s)$ has property A, then $a_s = \varphi(s)\mu(s)$ has the same property, where $\mu(s)$ is either any rational function having no poles in the half-plane $\operatorname{Re}(s) \geqslant 0$ and at infinity, or the Dirichlet series $\sum_{k=1}^{\infty} b_k k^{-s}$, absolutely convergent for $\operatorname{Re}(s) \geqslant 0$, or $\Gamma(\beta)\Gamma(s+\alpha)/\Gamma(s+\beta)\Gamma(\alpha)$, where $\operatorname{Re}(\beta) > \operatorname{Re}(\alpha) > 0$.*

Let us now consider the case where a_s has poles in the right half-plane. We shall write $f(x) \sim \varphi(x)$ if $f(x) = \varphi(x)(1+\eta)$, $\eta \to 0$ as $x \to \infty$.

Lemma. Suppose $0 < \operatorname{Re}(p) < 1$ or $p = 0$. If $\psi(x)$ has the properties: 1) in a neighborhood of $x = 0$ it has a convergent expansion of the form $\psi(x) = b_1 x + \cdots$; 2) it admits an analytic continuation to the half-axis $x \geqslant 0$ and

$$\psi(x) \sim Ax^p (\ln x)^{\alpha},$$

where A is an arbitrary complex number not equal to zero, and $\alpha \geqslant 0$ is an integer, then the function

$$\chi(x) = x^p \int_0^x t^{-1-p} \psi(t)\, dt$$

has the same properties 1), 2), and

$$\chi(x) \sim Ax^p \frac{(\ln x)^{\alpha+1}}{\alpha+1}.$$

Theorem 5. Let a_s be a rational function having poles in the half-plane $\mathrm{Re}(s) > 0$, and suppose that none of these poles has an integral abscissa, except possibly the point $s = 0$, which may be a pole. Denote by p the rightmost pole. Assume that on the line $\mathrm{Re}(s) = p$ the function a_s has no other poles.

Then the series (1) has an analytic continuation to the half-axis $x \geqslant 0$, and

$$f(x) \sim \begin{cases} c_{-m} \dfrac{(\ln x)^m}{m!}, & \text{in the case } p = 0, \\ c_{-m} \dfrac{\pi}{\sin p\pi} \dfrac{(\ln x)^{m-1}}{(m-1)!}\, x^p, & \text{in the case } \mathrm{Re}(p) > 0, \end{cases} \tag{4}$$

where m is the multiplicity of the pole $s = p$, and c_{-m} is the coefficient of $(s-p)^{-m}$ in the meromorphic part of a_s at this pole.

Proof. First consider the case $0 < \mathrm{Re}(p) < 1$. Let

$$\varphi_k(x) = \frac{x}{(1-p)^k} - \frac{x^2}{(2-p)^k} + \cdots, \qquad k = 0, 1, 2, \ldots,$$

a sequence of series with common radius of convergence $R = 1$.

For $0 \leqslant x < 1$,

$$\varphi_{k+1}(x) = x^p \int_0^x \frac{\varphi_k(t)}{t^{p+1}}\, dt, \qquad k = 0, 1, 2, \ldots.$$

For $k = 0$,

$$\varphi_1(x) = x^p \int_0^x \frac{dt}{t^p(1+t)} \sim \frac{\pi}{\sin p\pi}\, x^p.$$

Using the lemma, by induction we prove the assertion of the theorem for any integer $m \geqslant 1$. Passing from the first strip $0 < \mathrm{Re}(p) < 1$ to the second $1 < \mathrm{Re}(p) < 2$ is possible by considering $b_s = a_{s+1}$, and so on. The case of the pole $s = 0$ follows directly from the lemma.

It turns out that the assertion of Theorem 5 is valid not only for rational functions a_s. Thus, for example, the following holds:

Theorem 6. The asymptotic formulas (4) are valid for every function a_s of the form $a_s = r(s)\varphi(s)$, where $r(s)$ is a rational function satisfying the condition of Theorem 5, and $\varphi(s)$ belongs to one of the following types: 1) $1/\Gamma(s+1)$; 2) an entire function of first order of type $\sigma < \pi$; 3) the sum of a Dirichlet series absolutely convergent in the half-plane $\mathrm{Re}(s) \geqslant 0$; 4) $\Gamma(\beta)\Gamma(s+\alpha)/\Gamma(s+\beta)\Gamma(\alpha)$, where $\mathrm{Re}(\beta) > \mathrm{Re}(\alpha) > 0$ (p and c_{-m} with respect to a_s are understood in the same sense as in Theorem 5).

The proof is omitted for lack of space.

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Note: Figure translations are in progress. See original paper for figures.