

REMARKS ON THE MOTION OF BODIES WITH LARGE VELOCITIES IN A WEAK GRAVITATIONAL FIELD

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Abstract

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PHYSICS

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**REMARKS ON THE MOTION OF BODIES
WITH LARGE VELOCITIES IN A WEAK
GRAVITATIONAL FIELD**

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As an approximate representation of the motion of a continuous medium in its own weak gravitational field in the relativistic case, one may use the equations

$$\frac{\partial T_{ik}}{\partial x_k} = f_i, \quad (1)$$

where T_{ik} is the total energy-momentum tensor of the continuous medium and of the electromagnetic field;

$$f_i = -\frac{c^2}{2v} \frac{\partial \ln(1 + 2\varphi/c^2)}{\partial x_i} = -\frac{1}{v} \frac{\partial \psi}{\partial x_i} \quad (2)$$

is the force of gravity acting on the given element of the medium; φ is the potential of the gravitational field, which in our approximation we take to be not a tensor, as in the general theory of relativity, but a scalar; v is the specific volume; ψ is an auxiliary potential.

Neglecting the electromagnetic field, we write (1)

$$T_{ik} = (p + \varepsilon)u_i u_k + \delta_{ik}p. \quad (3)$$

Since in the case of an ordinary medium (especially a rarefied one) $p \ll \rho c^2 = \varepsilon$, we shall assume that the potential of the gravitational field satisfies the equation

$$\frac{\partial^2 \varphi}{\partial x_k^2} = -4\pi G \frac{\rho}{\theta^2}, \quad (4)$$

where G is the gravitational constant; $\theta^2 = 1 - a^2/c^2$; a is the velocity of motion of the gas. In equations (3) and (4), p is the pressure; u_i is the 4-velocity; ε is the energy density; ρ is the density of the medium.

In addition, it is necessary to know the equation of state of the medium, of the form

$$p = p(v; T) \quad \text{or} \quad p = p(\sigma; T), \quad (5)$$

where T is the temperature; σ is the entropy.

In the case of adiabatic flows possessing central symmetry, the basic system of equations takes the form

$$\begin{aligned} \frac{d(wu)}{ds} + \frac{\partial w}{\partial r} &= T \frac{\partial \sigma}{\partial r} - \frac{d(\psi u)}{ds}; & -\frac{\partial^2 \varphi}{c^2 \partial t^2} + \frac{\partial^2 \varphi}{\partial r^2} + \frac{N}{r} \frac{\partial \varphi}{\partial r} &= -4\pi G \frac{\rho}{\theta^2}; \\ \frac{d \ln v}{ds} &= \frac{\partial u_4}{\partial x_4} + \frac{\partial u}{\partial r} + \frac{Nu}{r} - \frac{1}{N + \psi} \frac{\partial \varphi}{\partial s}, \end{aligned} \quad (6)$$

where

$$w = pv + svc^2 - \psi \quad (7)$$

is the heat content; $N = 0, 1, 2$, respectively, for plane, cylindrical, and spherical waves; $ds = c\theta dt$.

Let us consider the following problem. Suppose that some volume of a medium with mass M_0 exploded with a large release of energy and began to expand, the energy density being so large that the peripheral part of the expanding gas could acquire velocities approaching the speed of light. After some time, when the pressure inside the expanding gas has fallen, one may consider the motion as occurring without the action of internal pressure in its own gravitational field. In those cases where the quantity $GM/ra^2 < 1$, where $\partial M/\partial r = 4\pi r^N/\theta^2$ and $p \rightarrow 0$, $T \rightarrow 0$, the system of equations (6), in the usual three-dimensional notation, takes the form

$$\begin{aligned} \frac{\partial a}{\partial t} + a \frac{\partial a}{\partial r} &= \theta^4 \frac{\partial \varphi}{\partial r}; & \frac{\partial \ln s}{\partial t} + a \frac{\partial \ln s}{\partial r} + \frac{1}{\theta^2} \left(\frac{\partial a}{\partial r} + \frac{a}{c^2} \frac{\partial a}{\partial t} \right) + \frac{Na}{r} &= 0; \\ \frac{\partial^2 \varphi}{\partial r^2} + \frac{N}{r} \frac{\partial \varphi}{\partial r} &= -4\pi G \frac{\rho}{\theta^2}. \end{aligned} \quad (8)$$

Here we neglect quantities proportional to φ/c^2 .

It is convenient to write this system, eliminating φ , in the form

$$\frac{\partial a}{\partial t} + a \frac{\partial a}{\partial r} = - \left(1 - \frac{a^2}{c^2}\right)^2 \frac{GM}{r^N}, \quad \frac{\partial M}{\partial t} + a \frac{\partial M}{\partial r} = 0. \quad (9)$$

If r, M are taken as the independent variables, then system (9) assumes the form

$$\frac{1}{2(1 - a^2/c^2)^2} \frac{\partial(a^2/c^2)}{\partial r} = - \frac{GM}{c^2 r^N}; \quad (10)$$

$$\frac{\partial t}{\partial r} = \frac{1}{a}. \quad (11)$$

Of greatest interest is the study of motion with spherical symmetry ($N = 2$). In this case integration gives

$$a^2 = a_0^2 \frac{1 + \frac{2\bar{R}}{r}}{1 + \frac{a_0^2}{c^2} \frac{2\bar{R}}{r}}, \quad (12)$$

where

$$\bar{R} = \frac{GM}{a_0^2} \left(1 - \frac{a_0^2}{c^2}\right) = R(M) = R(a_0).$$

Analysis of this equation shows that, depending on the value of a_0^2 , different trajectories are obtained.

Knowing at some initial instant of time $t = \bar{t}$, $a = \bar{a}$, $M = \bar{M}$, $r = \bar{r}$, we find

$$a_0^2 = \frac{\bar{a}^2 - \frac{2G\bar{M}}{\bar{r}} \left(1 - \frac{\bar{a}^2}{c^2}\right)}{1 - \frac{2G\bar{M}}{\bar{r}c^2} \left(1 - \frac{\bar{a}^2}{c^2}\right)}. \quad (13)$$

If $a_0^2 < 0$, then during expansion the particles stop at a finite distance (the elliptic case); if $a_0^2 = 0$, then the velocity of the particles becomes equal to zero at infinity (the parabolic case); if $a_0^2 > 0$, then the particles have a finite velocity at an infinitely large distance (the hyperbolic case).

Since $a_0 = a_0(M)$, depending on this function the following cases may occur: in an explosion all the gas remains at a finite distance and will fall toward the center, after which the expansion process will begin again, etc.; some particles

may go off to infinity, while others will take part in a pulsating process, remaining all the time at a finite distance; finally, with especially powerful release of explosion energy, all particles will tend to expand to infinity.

If the ratio GM/ra_0^2 is not very small, but the curvature of space may be neglected, then from (11) and (12) we have

$$t - t_0 = \int \frac{dr}{a} = -\frac{4\bar{R}}{a_0} \left(1 - \frac{a_0^2}{c^2}\right) \int \frac{d(a/a_0)}{(a^2/a_0^2 - 1)^2}, \quad (14)$$

where $t_0 = t_0(M)$, or

$$t - t_0 = -\frac{4R}{a_0} \int \frac{d(a/a_0)}{(a^2/a_0^2 - 1)^2} = -4GM \left(1 - \frac{a_0^2}{c^2}\right)^2 \int \frac{da}{(a^2 - a_0^2)^2} = F(a, a_0), \quad (15)$$

where $R = \bar{R}(1 - a_0^2/c^2)$, which differs from the classical case only in that before the integral sign there stands the constant factor $(1 - a_0^2/c^2)^2$.

Depending on the type of motion, the elementary integral (15) will lead either to periodic or to nonperiodic motions.

If $a_0^2 > 0$, then

$$t - t_0 = \frac{GM}{a_0^3} \left(1 - \frac{a_0^2}{c^2}\right)^2 \left[\frac{2aa_0}{a^2 - a_0^2} + \ln \frac{a + a_0}{a - a_0} \right]. \quad (16)$$

If $a_0^2 = 0$, then

$$t - t_0 = \frac{4}{3} \frac{GM}{a^3}. \quad (17)$$

If $a_0^2 < 0$, then

$$t - t_0 = \frac{2GM}{b_0^3} \left(1 + \frac{b_0^2}{c^2}\right)^2 \left[\frac{ab_0}{a^2 + b_0^2} + \text{arc tg } \frac{a}{b_0} \right], \quad (18)$$

where $b_0^2 = -a_0^2$.

Let us pass to the consideration of a concrete problem. Suppose that at some instant of time $t = 0$ an explosion occurred at the center of the mass with such a release of energy that the expansion became hyperbolic, and suppose that at some instant of time $t = \tau$, $r = r_0$, and the distribution $a_0 = a_0^*(r_0)$, $\rho_0 = \rho_0^*(r_0)$, $M = M^*(r_0)$ was established. Then

Fig. 1

Figure 1: Fig. 1

$$R = \frac{GM^*}{a_0^{*2}} \left(1 - \frac{a_0^*}{c^2}\right) = R^*(r_0); \quad t = \tau - F(a_0^*, a_0) = t_0^*. \quad (19)$$

Substituting the values $a = a_0^*$, $R = R^*$, $M = M^*$, $t_0 = t^*$ into (12) and into (16), (17), and (18), respectively, we find, eliminating r_0 , $a = a(r, t)$ for any instant of time $t \gg \tau$. For $a_0 = c$, $R = 0$, $a = a_0 = c$.

Let us consider the hyperbolic case for a weak field, when $GM/ra_0^2 \ll 1$. Obviously, in this case

$$a = a_0 \left(1 + \frac{R}{r}\right), \quad (20)$$

where

$$R = \frac{GM}{a_0^2} \left(1 - \frac{a_0^2}{c^2}\right) < r.$$

Next, we have

$$t - t_0 = \int \frac{dr}{a} = \frac{1}{a_0} \int \frac{dr}{1 + R/r} = \frac{r}{a_0} \left(1 + \frac{R}{r} \ln \frac{R}{r}\right), \quad (21)$$

where $t_0 = t_0(M)$. From equation (21) we find $a_0 = a_0(r, t)$, after which (20) gives

$$a = \frac{r}{t - t_0} \left(1 + \frac{R}{r}\right) \left(1 - \frac{R}{r} \ln \frac{r}{R}\right), \quad (22)$$

where $R = R(r, t)$.

Fig. 1

For the specific problem we shall have

$$\begin{aligned} a &= a_0^* \left(1 + \frac{R^*}{r}\right) = \\ &= \frac{r}{t - t_0^*} \left(1 + \frac{R^*}{r}\right) \left(1 - \frac{R^*}{r} \ln \frac{r}{R^*}\right). \end{aligned} \quad (23)$$

For $t > t_0^*$ the derivative is

$$\frac{\partial a}{\partial r} = \frac{1}{t} \left[1 - \frac{R^*}{r} + \frac{R^{*2}}{r^2} \left(\ln \frac{r}{R^*} - 1 \right) \right]. \quad (24)$$

For $R^*/r \ll 1$,

$$\frac{\partial a}{\partial r} = \frac{1}{t} \left[1 - \frac{R^*}{r} \right]. \quad (25)$$

It is evident that as r increases the quantity R^*/r decreases, the derivative $\partial a/\partial r$ increases and tends to the value $1/t$. The distribution of velocity with distance is shown schematically in Fig. 1 (curve 1). In the case where GM/ra_0^2 is not very small, there may be such an initial distribution of velocity over mass that, for $t > t_0^*$, the derivative $\partial a/\partial r$ will first decrease with distance, and then, as $a \rightarrow c$, increase again (Fig. 1, 2).

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CITED LITERATURE

1. L. D. Landau, E. M. Lifshitz, *The Theory of Fields*, § 14, 2nd ed., Moscow, 1948.

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