LOCAL REFLECTION IN WAVEGUIDES OF VARIABLE CROSS-SECTION

V. Pokrovskii, F. Ulinich, and S. Savvinykh

1958

SovietRxiv

Abstract

Full Text

LOCAL REFLECTION IN WAVEGUIDES OF VARIABLE CROSS-SECTION

V. Pokrovskii, F. Ulinich, and S. Savvinykh

(Presented by Academician M. A. Leontovich, 12 II 1958)

Physics

In the propagation of electromagnetic waves in waveguides of variable cross-section, scattering and reflection arise for three different reasons:

1. Scattering and reflection caused by local "defects" of the waveguide shape, i.e., corners, discontinuities of curvature, etc.

2. Reflection and scattering from cross-sections of critical dimensions for the given wave (turning points).

3. Nonlocal reflection caused by the irregularity of the waveguide shape as a whole.

In this article we shall consider local reflection and scattering of the first type. We shall assume the cross-sections of the waveguides to be constant at the ends and slowly varying in the transition region. For simplicity we shall illustrate the method of solution of the problem in the case of a plane waveguide of variable cross-section.

Let us direct the z-axis along the axis of the waveguide. We shall assume that the field does not depend on the coordinate x. We write the equation of the boundaries of the waveguide in the form

$$y = \pm f(\alpha z), \tag{1}$$

where α is a small parameter. As $z \to \pm\infty$, the function f tends, respectively, to the limits $f_{\pm}$. We emphasize that the total change of cross-section $f_{+} - f_{-}$ is not small.

Introduce an orthogonal coordinate system η, ζ in such a way that the lines $\eta = \pm 1$ coincide with the boundaries of the waveguide, while the coordinate ζ differs little from the coordinate z throughout the entire extent of the waveguide.

The fulfillment of these requirements is ensured by the following choice of the coordinates η, ζ:

$$y = \eta f(\alpha z), \qquad \frac{\alpha^2 y^2}{2} + \int_0^z \frac{f(\alpha z)}{f'(\alpha z)}\, dz = \int_0^\zeta \frac{f(\alpha\zeta)}{f'(\alpha\zeta)}\, d\zeta. \tag{2}$$

Solving these equations to accuracy up to α^3, we find

$$y = \eta f(\alpha\zeta)\left\{1 - \frac{\alpha^2\eta^2}{2}\,[f'(\alpha\zeta)]^2\right\}, \qquad z = \zeta - \frac{\alpha\eta^2}{2} f(\alpha\zeta) f'(\alpha\zeta). \tag{3}$$

It is known that the plane problem of electrodynamics reduces to solving the equation

$$\Delta U + k^2 U = 0 \tag{4}$$

with the boundary conditions $U\big|_\Sigma = 0$ for waves of electric type and $\left.\dfrac{\partial U}{\partial n}\right|_\Sigma = 0$ for waves of magnetic type.

In the coordinates η, ζ, equation (4) takes the form

$$\left(L_0 + \alpha^2 L_1 + \alpha^4 L_2 + ...\right) U + k^2 U = 0, \tag{5}$$

where

$$L_0 = \frac{1}{f^2}\frac{\partial^2}{\partial\eta^2} + \frac{1}{f}\frac{\partial}{\partial\zeta}\left(f\frac{\partial}{\partial\zeta}\right),$$

$$L_1 = \eta^2\left[(f')^2 + \frac{1}{2} f f''\right] L_0 + \frac{1}{f^2}\left[(f')^2 - \frac{1}{2} f f''\right]\frac{\partial}{\partial\eta}\left(\eta^2\frac{\partial}{\partial\eta}\right) - \frac{\eta^2}{f}\frac{\partial}{\partial\zeta}\left\{f\left[(f')^2 - \frac{1}{2} f f'\right]\frac{\partial}{\partial\zeta}\right\}. \tag{6}$$

The boundary conditions are written in the form

$$U\big|_{\eta=\pm 1} = 0 \quad \text{or} \quad \left.\frac{\partial U}{\partial\eta}\right|_{\eta=\pm 1} = 0.$$

Note that the operators L_0 and L_1 are terms of an expansion in powers of the parameter α^2.

We shall apply a method that is a combination of the Wentzel–Kramers–Brillouin method (the WKB method) and the ordinary perturbation method.

The solution of the equation of the zero approximation

$$\frac{1}{f^2}\frac{\partial^2 U_0}{\partial \eta^2} + \frac{1}{f}\frac{\partial}{\partial \zeta}\left(f\frac{\partial U_0}{\partial \zeta}\right) + k^2 U_0 = 0 \tag{7}$$

is sought in the form

$$U_0(\eta,\zeta) = \sum_{n=0}^{\infty}\left[U_{0n}(\zeta)\cos\lambda_n\eta + V_{0n}(\zeta)\sin\mu_n\eta\right], \tag{8}$$

where $\lambda_n = n\pi,\ \mu_n = (n+1/2)\pi$ for waves of electric type and $\lambda_n = (n+1/2)\pi,\ \mu_n = n\pi$ for waves of magnetic type. We shall consider here only symmetric types of waves.

For U_{0n} we obtain the equation

$$\frac{1}{f}\frac{d}{d\zeta}\left(f\frac{dU_{0n}}{d\zeta}\right) + k_n^2(\zeta)U_{0n} = 0, \qquad k_n^2 = k^2 - \frac{\lambda_n^2}{f^2}. \tag{9}$$

We seek the solution of equation (9) by the WKB method. In the zero approximation we obtain:

$$U_{0n}^{\pm} = \frac{A_n^{\pm}}{\sqrt{k_{nf}}}\exp\left[\pm i\int^{\zeta} k_n\, d\zeta'\right]. \tag{10}$$

We see that in the zero approximation the individual waves pass along the waveguide without scattering and without reflection. If the p-th derivative of the function f has a discontinuity of magnitude σ at the point ζ_0, then the solution of equation (9) will be represented as the sum of an incident and a reflected wave. The amplitude of the reflected wave A_n^- is related to the amplitude of the incident wave A_n^+ by the relation

$$A_n^- = \sigma\alpha^p\frac{2k^2A_n^+}{f}\left(\frac{i}{2k_n}\right)^{p+2}\exp\left[2i\int^{\zeta} k_n\, d\zeta'\right]\Bigg|_{\zeta=\zeta_0}. \tag{11}$$

Suppose that in the zero approximation only one wave with index l propagates from left to right:

$$U_0 = U_{0l}^{+}\cos\lambda_l\eta.$$

We seek the function of the first approximation U_1 in a form analogous to (8). For the functions $U_{1n}(\zeta)$ we obtain the equations

$$\frac{1}{f}\frac{d}{d\zeta}\left(f\frac{dU_{1n}}{d\zeta}\right) + k_n^2U_{1n} = R_{nl}U_{0l}^{+} + Q_{nl}\frac{dU_{0l}^{+}}{d\zeta}, \tag{12}$$

where

$$R_{nm} = \alpha_{nm}\left[((f')^2 + \frac{1}{2}ff'')k^2 - ((f')^2 - \frac{1}{2}ff'')k_m^2\right] - \beta_{nm}\frac{1}{f^2}\left((f')^2 - \frac{1}{2}ff''\right),$$

$$Q_{nm} = \alpha_{nm}\left((f')^2 - \frac{1}{2}ff''\right),$$

$$\alpha_{nm} = \alpha_{mn} = \int_{-1}^{1} \eta^2 \cos\lambda_n\eta \cos\lambda_m\eta \, d\eta,$$

$$\beta_{nm} = \beta_{mn} = \int_{-1}^{1} \frac{d}{d\eta}\left(\eta^2 \frac{d\cos\lambda_m\eta}{d\eta}\right)\cos\lambda_n\eta \, d\eta.$$

We express the solution of equation (12) by means of the Green function $G_n(\zeta, \zeta')$ of equation (9):

$$U_{1n} = -\int_{-\infty}^{\infty} G_n(\zeta,\zeta')\left[R_{nl}(\alpha\zeta')U_{0l}(\zeta') + Q_{nl}(\alpha\zeta')\frac{dU_{0l}}{d\zeta'}\right]d\zeta',$$

$$G_n(\zeta,\zeta') = \frac{i}{2}\frac{f(\alpha\zeta')\exp\left[i\ \mathrm{sgn}(\zeta-\zeta')\int_{\zeta'}^{\zeta} k_n\, d\zeta''\right]}{\sqrt{k_n(\alpha\zeta)\, k_n(\alpha\zeta')\, f(\alpha\zeta)\, f(\alpha\zeta')}}(1 + o(\alpha)). \tag{13}$$

Asymptotically, as $\zeta \to \pm\infty$, respectively, we obtain

$$U_{1n} = M_{nl}^{\pm}A_l^{+}\frac{\exp\left[\pm i\int^{\zeta} k_n\, d\zeta'\right]}{\sqrt{k_{n\pm}}\, f_{\pm}};$$

$$M_{nl}^{\pm} = -\frac{i}{2}\int_{-\infty}^{\infty}\frac{R_{nl} + ik_{lQ_{nl}}}{\sqrt{k_{nk}l}}\exp\left[i\int^{\zeta'}(k_l \mp k_n)\, d\zeta''\right]d\zeta'. \tag{14}$$

Under the assumption that α is small, the integrals $M_{nl}^{\pm}$ are approximately evaluated as

$$M_{nl}^{\pm} = -\alpha^{p-2}\frac{i^p\exp\left[i\int^{\zeta_0}(k_l \mp k_n)\, d\zeta'\right]}{(k_l \mp k_n)^{p-1}}\left.\frac{\sigma f\left[\alpha_{nl}(k^2 \pm k_{nk}l) + \beta_{nl}/f^2\right]}{4\sqrt{k_{nk}l}}\right|_{\zeta=\zeta_0}, \tag{15}$$

$$M_{ll}^{+} = -\frac{i}{2}\int_{-\infty}^{\infty} \frac{R_{ll}}{k_l}\, d\zeta.$$

We note that, in order to obtain the amplitude of the reflected wave with index l, one should add the quantities $M_{ll}^{-} A_l^{+}$ from (14) and A_l^{-} from (11).

The results obtained show that the effects of reflection and scattering depend substantially on the smoothness of the junction. The method developed above is readily generalized to the case of waveguides with similar cross sections, and can also be applied in the presence of turning points.

Institute of Radiophysics and Electronics
of the West Siberian Branch
of the Academy of Sciences of the USSR

Received
8 II 1958

Note: Figure translations are in progress. See original paper for figures.