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## Abstract

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## MATHEMATICS

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# SOME GENERAL THEOREMS ON THE BEHAVIOR OF SPECTRAL FUNCTIONS OF SECOND-ORDER DIFFERENTIAL SYSTEMS

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1. In this article we give two theorems on the asymptotic behavior, as $\lambda \to +\infty$, of the spectral functions of the differential system

$$-\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x) - \lambda\rho(x)y(x) = 0 \qquad (0 \le x < L \le \infty),$$

$$y(0) = n, \qquad p(x)\frac{d}{dx}y(x)\Big|_{x=0} = m, \tag{1}$$

where $m$ and $n$ $(m^2 + n^2 > 0)$ are real constants; $\lambda$ is a complex parameter; $\rho(x) \ge 0$, $p(x) > 0$, and $q(x)$ $(0 \le x < L)$ are real measurable functions such that for every $l \in (0, L)$

$$0 < \int_0^l \rho(x)\,dx < \infty, \qquad \int_0^l \frac{1}{p(x)}\,dx < \infty, \qquad \int_0^l |q(x)|\,dx < \infty.$$

Let us recall the definition of the spectral functions of system (1). Let $u(x;\lambda)$ be a solution of system (1). Denote by $M(x)$ the function defined by the equality

$$M(x) = \int_0^x \rho(s)\,ds.$$

As is known, a nondecreasing function $\tau(\lambda) = \tau(\lambda - 0)$ $(-\infty < \lambda < \infty;\ \tau(0) = 0)$ is called a **spectral function of system** (1) if, for every $M$-measurable function $f(x)$ $(0 \le x < L)$ having an $M$-summable square on $[0, L)$ and vanishing identically in some left neighborhood of the point $x = L$, the equality

$$\int_{-\infty}^{\infty}\left|\int_0^L f(x)u(x;\lambda)\,dM(x)\right|^2 d\tau(\lambda)=\int_0^L |f(x)|^2\,dM(x)\quad(<\infty)$$

holds.

2. We shall assign a nondecreasing function $\omega(\lambda)$ to the class $(K_\nu)$ if it is defined for $1\le\lambda<\infty$ (or on a wider set), $\omega(\lambda)\to\infty$ as $\lambda\to+\infty$, and there exist a number $\gamma<\nu$ and a sufficiently large number $N>1$ such that, for $\eta>\lambda>N$,

$$\frac{\omega(\eta)}{\omega(\lambda)}<\left(\frac{\eta}{\lambda}\right)^{\gamma}.$$

We shall assign a nondecreasing function $\theta(\lambda)$, defined for $1\le\lambda<\infty$, to– belong to the class $(\overline{K_\nu})$, if there exists a function $\omega(\lambda)\in(K_\nu)$ such that

$$\lim_{\lambda\to\infty}\frac{\theta(\lambda)}{\omega(\lambda)}=1.$$

It is obvious that, for any positive $\nu$, $(K_\nu)\subset(\overline{K_\nu})$, and for $\mu>\nu$, $(K_\mu)\supset(K_\nu)$, $(\overline{K_\mu})\supset(\overline{K_\nu})$.

In addition to the differential system (1), let us consider one more differential system

$$-\frac{d}{dx}\left(p_0(x)\frac{d}{dx}y(x)\right)+q_0(x)y(x)-\lambda\rho_0(x)y(x)=0\quad(0\le x<L_0\le\infty),$$

$$y(0)=n_0,\qquad p_0(x)\frac{d}{dx}y(x)\bigg|_{x=0}=m_0\tag{2}$$

of the same type as system (1).

**Theorem 1.** If $n=n_0\neq 0$,

$$\lim_{x\to\infty}\frac{p(x)}{p_0(x)}=1,\qquad\lim_{x\to 0}\frac{\rho(x)}{\rho_0(x)}=1\tag{3}$$

and at least one spectral function $\tau_0(\lambda)$ of system (2) belongs to the class $(\overline{K_1})$, then for any spectral function $\tau(\lambda)$ of system (1) (and, consequently, of system (2)) the equality

$$\lim_{\lambda\to\infty}\frac{\tau(\lambda)}{\tau_0(\lambda)}=1$$

holds.

Let us give an example. If $L_0=\infty$, $m_0=0$, $n_0=n\neq 0$, $\rho_0(x)=Sx^{\beta}$, $p_0(x)=Rx^{\alpha}$, $q_0(x)=0$ $(0\leq x<L_0)$, where $S>0$, $R>0$, $\beta>-1$, and $\alpha<1$, then the differential system (2) has a unique spectral function $\tau_0(\lambda)$, with $\tau_0(\lambda)=0$ for $\lambda<0$, while for $\lambda\geq 0$

$$\tau_0(\lambda)=n^{-2}S^{-\frac{1-\alpha}{\beta-\alpha+2}}R^{-\frac{\beta+1}{\beta-\alpha+2}}(1-\alpha)^{-\frac{\alpha+\beta}{\beta-\alpha+2}}T\left(\frac{\beta+\alpha}{1-\alpha}\right)\lambda^{\frac{\beta+1}{\beta-\alpha+2}},$$

where

$$T(\zeta)=(\zeta+2)^{-\frac{2(\zeta+1)}{\zeta+2}}(\zeta+1)\Gamma^{-2}\left(\frac{2\zeta+3}{\zeta+2}\right),\tag{4}$$

$\Gamma(z)$ is Euler’ s gamma function.

Since $\dfrac{\beta+1}{\beta-\alpha+2}<1$, in this case the function $\tau_0(\lambda)$ belongs to the class $(K_1)$ and, consequently, to the class $(\overline{K_1})$. Thus, with the choice indicated here of the functions $\rho_0(x)$, $p_0(x)$, and $q_0(x)$, and of the number $n_0$, system (2) satisfies the condition of the theorem. Therefore, if $n=0$, and

$$\lim_{x\to\infty}\rho(x)x^{-\beta}=S,\qquad \lim_{x\to 0}p(x)x^{-\alpha}=R\quad(\alpha<1;\ \beta>-1),\tag{5}$$

then for any spectral function of system (1), as $\lambda\to+\infty$, the following asymptotic equality holds:

$$\tau(\lambda)=n^{-2}S^{-\frac{1-\alpha}{\beta-\alpha+2}}R^{-\frac{\beta+1}{\beta-\alpha+2}}(1-\alpha)^{-\frac{\alpha+\beta}{\beta-\alpha+2}}T\left(\frac{\beta+\alpha}{1-\alpha}\right)\lambda^{\frac{\beta+1}{\beta-\alpha+2}}+O\left(\lambda^{\frac{\beta+1}{\beta-\alpha+2}}\right),$$

where $T(\zeta)$ is defined by equality (4).

Putting, in particular, $\alpha=\beta=0$ and $R=S=1$, we obtain that when $n=1$ and

$$\lim_{x\to\infty}\rho(x)=1,\qquad \lim_{x\to 0}p(x)=1,$$

for any spectral function $\tau(\lambda)$ of system (1), as $\lambda\to+\infty$ the asymptotic equality

$$\tau(\lambda) = \frac{2}{\pi}\sqrt{\lambda} + O(\sqrt{\lambda})$$

holds.

In the case when $p(x) \equiv 1$ and $\rho(x) \equiv 1$ $(0 \leq x < \infty)$, the last equality was first obtained by V. A. Marchenko ($^{6}$) and was subsequently refined more than once ($^{5,7}$).

For the case when $n = 0$, the following proposition holds.

**Theorem 2.** *Let* $n = n_0 = 0,\ m = m_0 \neq 0$, *let conditions (3) be satisfied, and let at least one spectral function* $\tau_0(\lambda)$ *of system (2) belong to the class* $(K_2)$*; furthermore, let the function* $\sigma_0(\lambda)$*, connected with* $\tau_0(\lambda)$ *by the equality*

$$\sigma_0(\lambda) = \int_1^{\lambda} \frac{d\tau(\xi)}{\xi} \qquad (\lambda > 1),$$

*belong to the class* $(\overline{K}_1)$*, and*

$$\lim_{\lambda\to\infty} \lambda\tau_0^{-1}(\lambda)\sigma(\lambda) < \infty.$$

*Then for every spectral function* $\tau(\lambda)$ *of system (1) (and, consequently, of system (2)) the equality*

$$\lim_{\lambda\to\infty} \tau(\lambda)/\tau_0(\lambda) = 1$$

*holds.*

In the case when $L_0 = \infty,\ m_0 = m \neq 0,\ n_0 = 0,\ \rho_0(x) = Sx^{\beta},\ p_0(x) = Rx^{\alpha}$, and $q_0(x) = 0$ $(0 \leq x < L)$, where $S > 0,\ R > 0,\ \beta > -1$ and $\alpha < 1$, the differential system (2) has the unique spectral function $\tau_0(\lambda)$:

$$\tau_0(\lambda) = \left[SR^{\frac{\beta+1}{1-\alpha}}(1-\alpha)^{\frac{\alpha+\beta}{1-\alpha}}\right]^{\frac{1-\alpha}{\beta-\alpha+2}} T_1\left(\frac{\alpha+\beta}{1-\alpha}\right)\lambda^{\frac{\beta-2\alpha+3}{\beta-\alpha+2}} \qquad (\lambda > 0),$$

where

$$T_1(\zeta) = (\zeta+2)^{-\frac{2}{\zeta+2}}(\zeta+3)^{-1}\Gamma^{-2}\left(\frac{\zeta+3}{\zeta+2}\right).$$

It is easy to see that in this case $\tau_0(\lambda)$ satisfies the condition of Theorem 2. Therefore, if $n = 0$ and conditions (5) are satisfied, then for any spectral function $\tau(\lambda)$ of system (1), as $\lambda \to +\infty$, the asymptotic equality

$$\tau(\lambda) = \left[SR^{\frac{\beta+1}{1-\alpha}}(1-\alpha)^{\frac{\alpha+\beta}{1-\alpha}}\right]^{\frac{1-\alpha}{\beta-\alpha+2}} T_1\left(\frac{\alpha+\beta}{1-\alpha}\right)\lambda^{\frac{\beta-2\alpha+3}{\beta-\alpha+2}} + o\left(\lambda^{\frac{\beta-2\alpha+3}{\beta-\alpha+2}}\right).$$

3. Theorems 1 and 2 also extend to the case when $\rho(x)$, $\dfrac{1}{p(x)}$, $q(x)$, $\rho_0(x)$, $\dfrac{1}{p_0(x)}$, and $q_0(x)$ are generalized derivatives of the functions $M(x)$, $N(x)$, $Q(x)$, $M_0(x)$, $N_0(x)$, and $Q_0(x)$, respectively, where the function $M_0(x)$ $(M(0) = 0)$ is nondecreasing on $[0, L_0)$; $N_0(x)$ $(N(0) = 0)$ is continuous and monotonically increasing on $[0, L_0)$; $Q_0(x)$ is real-valued and

having bounded variation on each interval $[0, l)$, where $0 < l < L_0$, while the functions $\hat{M}(x)$, $N(x)$, and $Q(x)$ satisfy analogous conditions on $[0, L)$. In addition, one must assume that $M(x) > M(+0) = M(0)$ and $M_0(x) > M_0(+0) = M(0)$ for $x > 0$, and replace conditions (3) by the conditions

$$\lim_{x,s\to 0}\frac{M(x)-M(s)}{M_0(x)-M_0(s)} = 1, \qquad \lim_{x,s\to 0}\frac{N(x)-N(s)}{N_0(x)-N_0(s)} = 1. \tag{6}$$

In this form, Theorems 1 and 2 are generalizations of propositions previously proved by the author (see [1], Theorems 3 and 4). This is easily verified if Theorems 1 and 2, in this generalized form, are applied to the case where $M_0(x) = (\beta+1)^{-1}x^{\beta+1}$ and $N_0(x) = x$.

Theorems 1 and 2 also admit a further generalization. Namely, if conditions (6) are not satisfied, but on a sufficiently small interval $[0, b]$ there exists a monotone continuous function $x(t)$ such that

$$\lim_{t,r\to 0}\frac{M(x(t))-M(x(r))}{M_0(t)-M_0(r)} = 1, \qquad \lim_{t,r\to 0}\frac{N(x(t))-N(x(r))}{N_0(t)-N_0(r)} = 1,$$

then, with all the other conditions retained, the assertions of Theorems 1 and 2 hold.

In proving the propositions presented in the present article, use was made of works of M. G. Krein ([2,3]), which give a description of the set of spectral functions of second-order differential systems, and of one general Tauberian theorem of B. I. Korenblum ([4]).

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*Note: Figure translations are in progress. See original paper for figures.*

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